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LONDON MATHEMATICAL SOCIETY.

FINANCIAL REPORT FOR THE SESSION 1883-84 (NOV. 9TH, 1883, TO NOV. 12TH, 1884).

CASH ACCOUNT.

Cr.

		£.	s.	d.
Balance from 1882-83:—				
General Fund	£238 5 3			
Life Composition Fund ...	10 10 0			
De Morgan Medal Fund...	7 17 1			
		256	12	4
Interest on Capital—				
Lord Rayleigh's Fund ...	54 4 8			
Life Composition Fund ...	20 8 1			
Invested Surplus Fund ...	4 10 4			
		79	3	1
18 Entrance Fees		18	18	0
101 Subscriptions—				
1 for 1880-81...	1 1 0			
8 for 1881-82...	8 8 0			
57 for 1882-83...	59 17 0			
33 for 1883-84...	34 13 0			
2 for 1884-85...	2 2 0			
		106	1	0
Sales of *Proceedings* and Extra Copies		65	1	0
De Morgan Medal Fund—				
Interest on Capital		3	2	10
8 Life Compositions		84	0	0
		£612	18	3

Dr.

		£.	s.	d.
Printing *Proceedings*, &c.		166	0	3
Purchase of Journals...		6	13	6
Binding		9	14	2
Postage, Stationery, Circulars, Bank Charges, and Sundries		14	11	2
Rent		10	0	0
Coals and Gas		1	10	0
Attendance—				
British Association... ...	£1 5 0			
Mr. Stewardson	5 5 0			
		6	10	0
Life Composition Fund—				
Purchase of £92. 15*s*. 4*d*. Three per Cent. Consols		94	10	0
Balance at Bank—				
General Fund	£292 9 3			
De Morgan Medal Fund...	10 19 11			
		303	9	2
		£612	18	3

Audited and found correct,

19*th November*, 1884. (Signed) P. A. MACMAHON.

ASSETS AND LIABILITIES.

General Fund.

			£.	s.	d.				£.	s.	d.
Cash at Bank			292	9	3	Subscriptions irrecoverable, to be struck off—					
101 Subscriptions due and owing—						1 for 1881–82	£1 1 0				
3 for 1880–81	£3 3 0					1 for 1882–83	1 1 0				
11 for 1881–82	11 11 0								2	2	0
13 for 1882–83	13 13 0					2 Subscriptions for 1884–85, paid in advance...			2	2	0
*74 for 1883–84	77 14 0					Balance			394	6	3
			106	1	0						
			£398	10	3				£398	10	3

* Viz., 111, the number of Subscribers for 1883–84, less 4 subscriptions paid in advance in 1882–83, and 33 paid during 1883–84.

De Morgan Medal Fund.

	£.	s.	d.		£.	s.	d.
Cash at Bank, the Proceeds of $3\frac{1}{2}$ years' Dividends on Invested Capital	10	19	11	Balance	10	19	11

CAPITAL ACCOUNT.

	Sum invested. £.	s.	d.		£.	s.	d.	Description of Investment.
General Fund—								
1. Life Composition Fund...	721	0	0	... ...	741	7	0	Three per Cent. Consols.
2. Lord Rayleigh's Fund	1000	0	0	... ...	870	0	0	Guaranteed Five per Cent. Great Indian Peninsular Railway Stock.
3. Invested Surplus Fund...	150	0	0	... ...	150	11	3	New Three per Cents.
De Morgan Medal Fund	103	5	3	... ...	104	19	8	Reduced Three per Cents.

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PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. XVI.

FROM NOVEMBER, 1884, TO NOVEMBER, 1885.

LONDON:

FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

LONDON:
PRINTED BY C. F. HODGSON AND SON,
GOUGH SQUARE, FLEET STREET.

PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. XVI.

TWENTY-FIRST SESSION, 1884-5.

November 13*th*, 1884.

ANNUAL GENERAL MEETING, held at 22 Albemarle Street, W.

Professor HENRICI, F.R.S., President, in the Chair.

Prof. Karl Pearson, M.A., University College, London, Fellow of King's College, Cambridge, was elected a Member.

The President then announced, in the following words, the losses the Society had sustained by death during the Recess :—

" Before our ordinary business begins, I have to address you once again,* to say a few words in memory of two of our Members, Prof. Rowe, of Cambridge, and Prof. Townsend, of Dublin, who have died since our last meeting.

" Prof. Rowe never enjoyed good health, and died when still a young man. I first saw him in the examination-room of the London University, when Prof. Townsend and I were examiners. We were both struck by the brilliancy of his answers, but shocked by the extreme delicacy of his appearance. Later on, when he became my colleague at University College, this impression was somewhat effaced by his vivacity. He was a man of great amiability and geniality, and these qualities, together with his great musical gifts, made him friends

* [Alluding to the previous announcements of the deaths of Mr. Merrifield and Dr. Todhunter during the President's tenure of office.]

wherever he went. The principal thing he has left us as a mathematician is his paper in the *Phil. Trans.* of the Royal Society, on Abelian Functions, which showed that we might have expected good work from him, had he lived.

"Prof. Townsend had come to ripe years, though he was by no means an old man. He, like Prof. Rowe, was characterised by great geniality. He was beloved by his numerous pupils, and held in the greatest esteem by all who knew him intimately for his kindness and warm-heartedness. As a teacher he is said to have possessed singular qualifications.

"Personally, I have had the privilege of working with him for four years as examiner at the London University, and learned to value him very highly. As a mathematician, he was devoted to Pure Geometry, though he has published many papers belonging to Applied Mathematics. He treated his subject with considerable elegance. His chief work is his 'Chapters on the Modern Geometry of the Point, Line, and Circle,' which, though they are confined to a somewhat narrow field, are very rich in methods, and many of the results can at once be extended to Conics in general. He had just obtained a Senior Fellowship at Trinity College, Dublin, when the illness began which caused his death."

After a slight pause, he presented the De Morgan Memorial Gold Medal to Prof. Cayley, accompanying the presentation with the following address:—

"You will remember that, two years ago, it was announced from this chair that the Council had settled the conditions under which the De Morgan Medal should be given, and that the first award would be made at the anniversary meeting of 1884.

"I have now to make the announcement that the Council has decided that the first medal should be given to Professor Cayley in acknowledgment of his work in the 'Theory of Invariants.'

"As this is the first award of the medal, I may remind you of its origin. Soon after the death of De Morgan, some of his admirers started a subscription, for the double purpose of having a bust executed and founding a medal to be given in his memory. The *Bust* now adorns the Library of the London University, where also his valuable collection of books is preserved. The *Medal* was offered to the Mathematical Society, and its Council accepted the honourable duty of determining its award. There is a peculiar fitness in the medal being thus connected with our Society; for this Society was founded with the active co-operation of De Morgan by a number of his advanced students, among whom his talented son George, who

died soon afterwards, took the lead. De Morgan himself was the first President, and our *Proceedings* began with a very characteristic opening speech by him.

"The Medal is to be given for eminent original work in Mathematics, and no more fitting memorial than this could in my opinion be devised for a man who spent his whole life in carefully preparing the foundation for such work by his teaching and his writings.

"De Morgan was pre-eminently a teacher. His most original work does not so much increase our stock of mathematical knowledge, but is concerned with mathematical reasoning, and with exact reasoning in general.

"In the opening speech referred to, De Morgan himself divides exact science into two branches, *the analysis of the necessary laws of thought*, and *the analysis of the necessary matter of thought*. His own work belongs to the former. He was a logician much more than a mathematician in the ordinary sense of the word, and, when reading his mathematical works, I have always had the feeling that he studied Mathematics not so much for its own sake as on account of the logic contained and exemplified in it. I once made this remark in the Professors' Common Room of University College, when an old colleague of his turned round and said, 'You are quite right; he told me so himself.'

"In this work De Morgan did not stand alone; we may almost take him as a type of his period. It has often struck me as a noteworthy fact that in England, after a long pause in mathematical activity, the work taken first in hand was investigation into the very bases of Mathematics, and more particularly into mathematical reasoning. These investigations became partly a mathematical analysis of Logic itself, and partly a logical analysis of the laws followed by the symbols and operations used in Mathematics. De Morgan worked in both directions; we have his 'Formal Logic' and his 'Double Algebra.' Operations were studied quite independently of the meaning given to the symbols. Originally, the symbols stood for concrete things, and each operation had its concrete meaning. At present, symbols are sometimes used without giving them any meaning whatsoever, and without defining them at all; and then the operations for combining these symbols are arbitrarily defined, with the sole restriction that they do not contradict each other.

"Each new set of operations thus establishes a calculus. If afterwards any entities can be found which can be combined by operations answering the characteristics of the operations used in the new calculus, then the latter may be employed for a theory of those entities, and its results will allow of an interpretation. These entities

themselves may be anything—concrete things, or logical concepts, or ordinary algebraical quantities.

"Thus the ground was already prepared for greatly extending the realm of Algebra, and the scope and power of algebraical operations, when the genius of Prof. Cayley conceived the idea of Invariants, which has given rise to that marvellous growth of our science which has suddenly brought England again far to the front.

"It was known from Gauss's investigations that for Quadratic expressions a certain combination of its constants, its determinant, exists which has the following property:—

"If the Quadratic expression be transformed into another by a linear substitution, then the determinant of the transformed expression is obtained from that of the original expression by multiplying it by a factor which depends solely on the substitution used.

"Afterwards Eisenstein discovered that a similar theorem holds for a cubic expression of one variable. These isolated facts suggested to Cayley that combinations of constants having this property must exist for all algebraical expressions. The problem was, how to find these.

"The manner in which this has been solved I need not restate here, but I wish to call your attention to the fact that the symbolic methods worked out by the school of mathematicians referred to have been of the greatest use in the development of the Theory of Invariants, which could scarcely have been brought to its present perfection without it.

"It would be an impertinence for me to say much either in praise of Prof. Cayley's work or in justification of the Council's choice. Prof. Cayley has invented and worked out the Theory of Invariants, and, in steady life-long work, connected it with nearly every branch of Mathematics, enriching everything he touches, and everywhere throwing open new vistas of future work.

"The Council of the Mathematical Society, in selecting Prof. Cayley as the first recipient of the De Morgan Medal, and thus doing homage to his genius, did so not so much with the idea that it could add honour to his name, as that they might add honour to the Medal, by connecting his great name with it, and thus increase its value for all future recipients. And it is befitting that a body like the London Mathematical Society should give formal expression to the reverence and admiration in which it holds the greatest among its members.

"I shall now have the honour, the greatest which has ever fallen to me, of handing this medal to Prof. Cayley, and I call upon him to receive it.

"Professor Cayley, I hand this medal over to you, in the name of

the London Mathematical Society, as a token of their respect and admiration."

Prof. Cayley briefly returned thanks, and waived all claim to priority of discovery of Invariants.*

The Treasurer (Mr. A. B. Kempe) read his report. Its reception was moved by Mr. S. Roberts, seconded by Prof. Greenhill, and carried unanimously.

At the request of the Chairman, Captain P. A. Macmahon, R.A., consented to act as Auditor.

From the Report of the Secretaries, it appeared that the number of members since the last General Meeting, held November 8th, 1883, had increased from 159 to 173, of these 61 being Life Members.

The Obituary of the Society comprised the following names :—

Dr. Isaac Todhunter, F.R.S., elected June 18th, 1865, died March 1st, 1884.

C. W. Merrifield, F.R.S., elected March 19th, 1866, died January 1st, 1884.

Rev. R. Townsend, F.R.S., elected April 16th, 1866, died October 17th, 1884.

R. C. Rowe, M.A., elected November 13th, 1879, died September 21st, 1884.

The following communications had been made :—

Symmetric Functions, and in particular on certain Inverse Operators in connection therewith: Captain P. A. Macmahon, R.A.

On a Certain Envelope: Prof. Wolstenholme, D.Sc.

On certain results obtained by means of the Arguments of Points on a Plane Curve: R. A. Roberts, M.A.

Multiple Frullanian Integrals, Part iii.: E. B. Elliott, M.A.

Note on Jacobi's Transformation of Elliptic Functions: J. Griffiths, M.A.

Symmedians and the Triplicate-Ratio Circle: R. Tucker, M.A.

The Form of Standing Waves on the surface of Running Water: Lord Rayleigh, F.R.S.

A Method of finding the Plane Sections of a Surface, and some Considerations as to its extension to Space of more than Three Dimensions: W. J. C. Sharp, M.A.

On a Deduction from the Elliptic-Integral Formula

$$y = \sin(A+B+C+\dots):$$

J. Griffiths, M.A.

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* [*Cf.* Dr. Salmon's "Lessons on Higher Algebra," pp. 103, 295.]

An Extension of Pascal's Theorem to Space of Three Dimensions; and on the Theory of Screws in Elliptic Space: A. Buchheim, B.A.

On Contacts and Isolations, a Problem in Permutations: H. Fortey, M.A.

On the Induction of Electric Currents in Cylindrical and Spherical Conductors: Prof. H. Lamb, M.A.

On a Group of Circles which are connected with the Triplicate-Ratio Circle: R. Tucker, M.A.

On the Intersections of a Triangle with a Circle: H. M. Taylor, M.A.

On the Function which denotes the difference between the number of $(4m+1)$-divisors and the number of $(4m+3)$-divisors of a Number: J. W. L. Glaisher, F.R.S.

On a General Theory including the Theories of Systems of Complexes and Spheres: A. Buchheim, B.A.

On the Square of Euler's Series: J. W. L. Glaisher, F.R.S.

Further Results from a Theory of Transformation of Elliptic Functions: J. Griffiths, M.A.

Note concerning the Pellian Equation: S. Roberts, F.R.S.

On the closed Link Polygons belonging to a system of Coplanar Forces having a Single Resultant: Prof. M. J. M. Hill, M.A.

On the direct Application of the Principle of Least Action to Dynamical Analogues, Parts i., ii.: Prof. J. Larmor, M.A.

On Double Algebra: Prof. Cayley, F.R.S.

On Direct Investigation of the Complete Primitive of the Equation $F(x, y, z, p, q) = 0$, with a way of remembering the Auxiliary System: J. W. Russell, M.A.

On Electrical Oscillations and the effects produced by the Motion of an Electrified Sphere: J. J. Thomson, F.R.S.

Motion of a Network of Particles, with some analogies to Conjugate Functions: E. J. Routh, D.Sc., F.R.S.

On a Subsidiary Elliptic Function $pm\,(u, k)$: J. Griffiths, M.A.

On the Homogeneous Equation of a Plane Section of a Geometrical Surface: J. J. Walker, F.R.S.

On a Birational Transformation of Space of Three Dimensions, of the Sixth Degree, the Inverse of which is of the Fifth: Prof. Cremona, F.R.S.

Some Properties of Two Lines in the Plane of a Triangle: R. Tucker, M.A.

Note on the Induction of Electric Currents in a Cylinder placed across the Lines of Magnetic Force: Prof. H. Lamb, M.A.

Minor communications were made by Prof. Sylvester, F.R.S., and J. Hammond, M.A.

Additional exchanges of *Proceedings* were made with Dr. Bierens de Haan (*Nieuw Archief vur Wiskunde*), and Prof. Liguine (Mathematical Society of Odessa)*.

The same journals had been subscribed for as in the preceding Session.

The meeting then proceeded to the election of the new Council. The Scrutators (Mr. G. Heppel and Prof. M. J. M. Hill) having examined the Balloting Lists, declared the following gentlemen duly elected:—

President, J. W. L. Glaisher, F.R.S.; Vice-Presidents, Dr. Henrici, F.R.S., Prof. Sylvester, F.R.S., J. J. Walker, F.R.S.; Treasurer, A. B. Kempe, F.R.S.; Hon. Secs., M. Jenkins, M.A., and R. Tucker, M.A.; other members, Prof. Cayley, F.R.S., Sir J. Cockle, F.R.S., E. B. Elliott, M.A., Prof. Greenhill, M.A., J. Hammond, M.A., Prof. H. Hart, M.A., Dr. Hirst, F.R.S., S. Roberts, F.R.S., R. F. Scott, M.A.

Dr. Henrici having thanked the members for "the kind indulgence they had shown him" during his Presidency, Mr. J. W. L. Glaisher then took the chair, and thanked the Society for the high honour they had conferred upon him.

Mr. Tucker read abstracts of the following papers:—

On the Theory of Screws in Elliptic Space (Supplementary Note), and on the Theory of Matrices: Mr. A. Buchheim.

On Sphero-Cyclides: Mr. H. M. Jeffery.

Results from a Theory of Transformation of Elliptic Functions: Mr. J. Griffiths.

On the Limits of Multiple Integrals: Mr. H. MacColl.

On the Motion of a Viscous Fluid contained in a Spherical Vessel: Prof. H. Lamb.

On Certain Conics connected with a Plane Unicursal Quartic: Mr. R. A. Roberts.

Note on Elliptic Functions, on an Integral Transformation, and a Theorem in Plane Conics: Mr. A. Mukhopâdhyây.

The President (Dr. Henrici taking the Chair) then read a paper on Certain Systems of q-series in Elliptic Functions in which the Exponents in the Numerators and in the Denominators are connected by Recurring Relations.

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* For list of exchanges, see Vol. xiv., p. 315.

The following presents were received:—

"Educational Times," for November.

"Scientific Transactions of the Royal Dublin Society" (Series II.), Vol. I., Parts XX.—XXV.; Vol. III., Parts I.—III.

"Scientific Proceedings of the Royal Dublin Society" (New Series), Vol. III., Vol. VI., VII.; Vol. IV., Parts I.—IV.

"Smithsonian Report for 1882," 8vo; Washington, 1884.

"Bulletin des Sciences Mathématiques et Astronomiques," October and November.

"Beiblätter zu den Annalen der Physik und Chemie," Band VIII., No. 10, 1884.

"Archives Néerlandaises des Sciences Exactes et Naturelles," T. XIX., L. 3; Harlem, 1884.

"Jahrbuch über die Fortschritte der Mathematik," XIV. 1, Jahrgang 1882; Berlin, 1884.

"Cours de Mécanique," par M. Despeyrous, avec des Notes par M. G. Darboux, Tome I., 8vo; Paris, 1884.

"Crelle," Bd. XCVII., Heft 3.

M. E. Lemoine presented the following pamphlets:—

Association Française pour l'Avancement des Sciences:

Congrès de Lyon, 1873—"Sur quelques propriétés d'un point remarquable d'un triangle."

Congrès de Lille, 1874—"Note sur les propriétés du centre des médianes antiparallèles dans un triangle." (*Suite.*)

Congrès de Nantes, 1875—"Note sur le tetraèdre dont les arêtes opposées sont égales deux à deux."

Congrès de Reims, 1880—"Questions de Probabilités et valeurs relatives des pièces du Jeu des Echecs."

Congrès d'Alger, 1881—"Quelques questions de géométrie de position sur les figures qui peuvent se tracer d'un seul trait."

Congrès de Rouen, 1883—"Sur les quatre groupes de deux points d'un triangle ABC qui sont en même temps les foyers d'une conique inscrite et d'une conique circonscrite à ce triangle," et "Sur les nombres formés des mêmes chiffres écrits en sens inverse."

"Bullétin de la Société Mathématique de France," 1883—"Quelques questions de Probabilités résolues géométriquemont."

"Journal de Mathématiques spéciales," 1883—"Nouveaux points remarquables du plan d'un triangle."

"Mémoires de la Société des Ingénieurs civils"—"Note sur le losange articulé du Colonel Peaucellier."

"Nouvelles Annales de Mathématiques," 1884—"Sur une question de Probabilité."

"Comptes Rendus," Oct. 1882—"Décomposition d'un nombre entier N en ses puissances $n^{\text{ièmes}}$ maxima."

On Certain Conics connected with a Plane Unicursal Quartic.

By R. A. ROBERTS, M.A.

[*Read Nov.* 13*th*, 1884.]

1. The object of the present paper is to demonstrate the existence of conics which are related to a unicursal quartic in such a manner that an infinite number of closed polygons can be simultaneously circumscribed about them and inscribed in the quartic. I have found that certain conics touching two double tangents and the curve twice, and certain other conics touching the curve four times, are connected with the quartic in this manner, if a further condition is satisfied. This condition depends on the number n of the sides of the polygon, and will give a finite number of conics for each value of n.

2. Suppose a conic U referred to two tangents x, y, and z, their chord of contact, to be written in the form $z^2-4xy=0$, then any tangent will be $\mu^2x-\mu z+y=0$, and, if μ_1, μ_2 are the parameters of the tangents passing through the point x, y, z, we may write

$$\rho x = 1, \quad \rho y = \mu_1\mu_2, \quad \rho z = \mu_1+\mu_2 \quad \text{..................(1)};$$

and hence we may consider μ_1, μ_2 as a system of coordinates determining the position of any point on the plane. This system has been used by Darboux ("Sur une classe remarquable de courbes et de surfaces algébriques," p. 183) in investigating some properties of polygons circumscribed about a conic and inscribed in another curve. Now, if we write $f(\mu_1)$, $f(\mu_2)$, where f is an arbitrary function, for μ_1 and μ_2 respectively, we have a transformation by which a given curve will be transformed into another, and, in particular, any tangent to U will be transformed into the product of as many tangents to U as the equation $f(\mu) =$ a constant has roots.

3. Let us consider now the quadratic transformation

$$f(\mu) = \frac{a+b\mu+c\mu^2}{a'+b'\mu+c'\mu^2} \quad \text{......................(2)},$$

then it is evident from (1) that the coordinates of any point on the transformed curve are expressed as functions of the second degree of those of the point on the given curve; and, hence, the transformed curve will be, in general, of double the degree of that of the given curve, and will have the same deficiency. Now, it is to be ob-

served that, if we substitute a homographic transformation for μ, this is only equivalent to a change of the axes of coordinates. It will follow hence that the transformation (2) is reducible to a transformation in which we substitute their squares for μ_1, μ_2, respectively. In this case, if α, β, γ is a point on the given curve, and x, y, z the corresponding point on the transformed curve, we may write

$$\rho x = \alpha^2, \quad \rho y = \beta^2, \quad \rho z = \gamma^2 - 2\alpha\beta \quad \text{..................(3).}$$

4. If we suppose the point α, β, γ to lie on a general conic V, α, β, γ can be expressed as quadratic functions of a parameter, and x, y, z will then lie on a unicursal quartic. Since the conics U and V and the point xy involve twelve constants, and a unicursal quartic involves eleven, it follows that, when the quartic is given, the conic U involves but one indeterminate constant, and, therefore, satisfies four conditions. It is easy to find what these conditions are; for from (3) we see that x, y, which are tangents to U, are also double tangents to the quartic, and, since we have

$$\rho^2 (z^2 - 4xy) = \rho^2 U = \gamma^2 (\gamma^2 - 4\alpha\beta),$$

it follows that U has double contact with the curve. Now, we have seen that to a polygon circumscribed about U corresponds a polygon also circumscribed about U, and, if the former polygon be inscribed in V, the latter polygon will be inscribed in the quartic; and we know that, in order that this should be possible, it is only necessary that U and V should be connected by a single relation. This relation, which depends upon the number of sides of the polygon, will evidently serve to determine a finite number of conics, such as U, connected with the quartic.

5. We can verify from (3) that a tangent to U meets the quartic in a pair of points whose parameters are connected by a relation of the same nature as that which connects the pair of points where the tangent meets V. Let a tangent to U be written $\mu^2 x - \mu z + y = 0$, then, for the four points where it meets the quartic, we have

$$\mu^2\alpha^2 + \beta^2 - \mu (\gamma^2 - 2\alpha\beta) = 0,$$

which breaks up into the two quadratics

$$\mu\alpha + \beta \pm \sqrt{\mu}\, \gamma = 0;$$

but these are evidently the equations which we should get to determine the points where the tangents $\mu x \mp \sqrt{\mu}\, z + y = 0$ meet V. If ϑ is the parameter involved in α, β, γ, this relation connecting either

pair of points may be written in a form involving elliptic integrals as follows:
$$\frac{d\vartheta_1}{\sqrt{(\gamma_1^2-4\alpha_1\beta_1)}} \pm \frac{d\vartheta_2}{\sqrt{(\gamma_2^2-4\alpha_2\beta_2)}} = 0.$$

6. Since a unicursal quartic has four double tangents, there are six pairs of such lines, and thus we have six distinct systems of conics of the kind found above. To investigate the conditions corresponding to a given number of sides of the polygon, I write the equation of the quartic in the symmetrical form
$$\sqrt{x}+\sqrt{y}+\sqrt{z}+\sqrt{u}=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(4),$$
where x, y, z, u are the four double tangents, and are connected by the identical relation
$$ax+by+cz+du=0.$$

When the curve is written in this form (4), it is easy to see that
$$\vartheta^2x+\vartheta\,(x+y-z-u-2\sqrt{zu})+y=0 \quad\ldots\ldots\ldots\ldots(5)$$
represents a conic touching the lines z, u and having double contact with the curve, and belongs to the system we have been considering. It may be observed that we can show, from (5), that the chord of contact with the curve is $y-\vartheta^2x=0$, which therefore passes through the fixed point xy; also, that the chord of contact with the double tangents is
$$\vartheta^2x+\vartheta\,(x+y-z-u)+y=0,$$
which touches the conic
$$(x+y-z-u)^2=4xy.$$

Putting now
$$\vartheta x+\frac{y}{\vartheta}+x+y-z-u=v,$$
the equation (5) becomes
$$v^2-4zu=0,$$
and this coincides with the conic U. By means, then, of the substitutions
$$z^2=z',\quad u^2=u',\quad v^2-2zu=v' \quad\ldots\ldots\ldots\ldots\ldots(6),$$
[see (3),] we must be able to transform the quartic
$$\sqrt{x'}+\sqrt{y'}+\sqrt{z'}+\sqrt{u'}=0$$
back into the conic V. From (6), we get
$$v^2-2zu=\vartheta x'+\frac{y'}{\vartheta}+x'+y'-(z^2+u^2),$$

and $$\sqrt{x'}+\sqrt{y'}=z-u;$$

hence $$v^2=\vartheta x'+\frac{y'}{\vartheta}+x'+y'-(\sqrt{x'}+\sqrt{y'})^2,$$

and $$v=\sqrt{(\vartheta x')}-\sqrt{\left(\frac{y'}{\vartheta}\right)}.$$

We thus find

$$v+\frac{(z-u)}{\sqrt{\vartheta}}=m\sqrt{x'},\quad v-(z-u)\sqrt{\vartheta}=-m\sqrt{y'};$$

but we have $$ax'+by'+cz'+du'=0;$$

hence we get, finally,

$$V\equiv\left(\frac{a+b\vartheta^2}{\vartheta}+cm^2\right)z^2+\left(\frac{a+b\vartheta^2}{\vartheta}+dm^2\right)u^2$$
$$+(a+b)\,v^2+\frac{2\,(a-b\vartheta)}{\sqrt{\vartheta}}\,v\,(z-u)-\frac{2\,(a+b\vartheta^2)}{\vartheta}\,zu=0,$$

where $m=\dfrac{1+\vartheta}{\sqrt{\vartheta}}$. Forming now the discriminant Δ of $V+\lambda U$, we get Δ proportional to

$$4\lambda^3+4\lambda^2\,(a+b\vartheta)\left(\frac{1+\vartheta}{\vartheta}\right)-\lambda\,\frac{(1+\vartheta)^2}{\vartheta^2}\{cd\,(1+\vartheta)^2+(c+d)(a+b\vartheta^2)-4ab\vartheta\}$$
$$-\frac{(1+\vartheta)^4}{\vartheta^3}\,(abc+bcd+cda+dab).$$

We may simplify this expression by putting

$$\lambda=\frac{(1+\vartheta)}{\vartheta}\,k,$$

when Δ is found to vary as

$$4k^3+4\,(a+b\vartheta)\,k^2-\{(cd+bc+bd)\,\vartheta^2+2\,(cd-2ab)\,\vartheta+cd+ac+ad\}\,k$$
$$-\vartheta\,(1+\vartheta)(abc+bcd+cda+dab)\;\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

Now, when the discriminant of $V+\lambda U$ is written in the form

$$A\lambda^3+B\lambda^2+C\lambda+D,$$

the condition that it should be possible to circumscribe a triangle about U which shall be inscribed in V is $B^2-4AC=0$. Hence, in this case, we get from (7), to determine ϑ,

$$(b+c)(b+d)\,\vartheta^2+2\,(cd-ab)\,\vartheta+(a+c)(a+d)=0\;\;\ldots\ldots(8).$$

There are, therefore, two conics of each of the six systems which touch the sides of an infinite number of triangles inscribed in the quartic. For the quadrilateral, we have

$$B^3-4ABC+8DA^2=0,$$

and we get then from (7), to determine ϑ,

$$(a+b\vartheta)^3+(a+b\vartheta)\{(cd+bc+bd)\,\vartheta^2+2\,(cd-2ab)\,\vartheta+cd+ac+ad\}$$
$$-2\vartheta\,(1+\vartheta)(abc+bcd+cda+dab)=0 \quad\text{..............(9).}$$

From the general conditions given by Professor Cayley, we find that for n, an odd number, $=2m+1$, the degree of the equation for ϑ is $2\,.\,4\,.\,6\ldots 2m$; and for n, an even number, $=2m$, is $1\,.\,3\,.\,5\ldots 2m-1$.

7. In a paper published by me in the *Proceedings*, "On Certain Results obtained by means of the Arguments of Points on a Plane Curve," Vol. xv., p. 4, I have shown, in § 17, that the tangents to a certain system of conics touching a bicircular quartic four times, meet the curve in pairs of points whose parameters are connected by the relations $\quad u_1+u_2=\sigma,\quad u_3+u_4=w-\sigma$ ....(10),

where σ depends on the undetermined constant in the equation of the conic, and w is an absolute constant of the curve. These relations (8) will enable us to determine conics of the system which touch the sides of an infinite number of polygons of an even number of sides inscribed in the curve. As, for instance, for a quadrilateral, if a, b, c, d are the arguments of the vertices, we have

$$a+b=\sigma,\quad b+c=w-\sigma,\quad c+d=\sigma,\quad d+a=w-\sigma,$$

whence, to determine σ, we get $4\sigma=2w$. The argument u, which for the binodal quartic is an elliptic integral, reduces for the trinodal quartic to a circular function or a logarithm. I proceed to investigate this result independently for the trinodal quartic. Let α, β, γ be quadratic functions of a parameter ϑ, then the equations

$$\rho x=\alpha\gamma,\quad \rho y=\beta\gamma,\quad \rho z=\gamma^2+\alpha\beta \quad\text{.................(11)}$$

represent a unicursal quartic which the conic $z^2-4xy\equiv U$, say, touches four times. Seeking, then, the points where the tangent to U,

$$\mu^2x-\mu z+y=0,$$

meets the curve, we get, from (11),

$$\mu^2\alpha\gamma+\beta\gamma-\mu\,(\gamma^2+\alpha\beta)=0,$$

which breaks up into the factors

$$(\gamma-\mu\alpha)(\mu\gamma-\beta) = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12),$$

showing that the parameters of the corresponding pairs of points belong to systems in involution. If we suppose the roots of γ to be $0, \infty$, and put $\alpha = \vartheta^2+p\vartheta+q,\ \beta = \vartheta^2+p'\vartheta+q'$,

we get, from $\gamma-\mu\alpha = 0$, $\qquad \vartheta_1\vartheta_2 = q$,

and from $\mu\gamma-\beta = 0$, $\qquad \vartheta_3\vartheta_4 = q'$.

Now, if any line $lx+my+nz = 0$, meet the curve, we have from (11),

$$\vartheta\,(l\alpha+m\beta)+n\,(\vartheta^2+\alpha\beta) = 0,$$

from which we see that the four parameters are connected by the relation $\vartheta_1\vartheta_2\vartheta_3\vartheta_4 = qq'$. We see thus that qq' is a constant belonging to the curve and independent of the particular conic, and may therefore be put equal to unity. Hence, for a polygon of $2n$ sides, we put

$$\vartheta_1\vartheta_2 = q,\quad \vartheta_2\vartheta_3 = \frac{1}{q},\ \ldots\ \vartheta_{2n}\vartheta_1 = \frac{1}{q},$$

from which we get $q^{2n}-1 = 0$. The factor q^2-1 is irrelevant, and, as q and $\frac{1}{q}$ lead to the same conic, there would appear to be $n-1$ solutions. We must, however, diminish this number by the solutions corresponding to a lesser number of sides of the polygon, if there are primes contained in n. Thus, for $n = 2$, there is but one conic, and for $n = 3$, there are two conics. Also for $n = 4$ and 5, there are two and four conics respectively.

We can easily show that there are three distinct systems of conics of the kind which we have been just considering. For, from (11), we see that xy is a node whose parameters are given by $\gamma = 0$, and, from (12), γ belongs to both the involutions. There are, therefore, three systems corresponding to the three pairs of nodes.

8. The conics of the two different kinds mentioned in § 1 are the only systems I have been able to discover that touch the sides of an infinite number of polygons inscribed in the general unicursal quartic. For unicursal quartics, however, satisfying certain invariant conditions, other conics can be found. For instance, if the quartic can be written in the form

$$x^2y^2-z^2uv = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13),$$

the conic touching the lines x, y, z, u, v will touch the sides of an in-

finite number of quadrilaterals inscribed in the curve (see p. 186, Darboux, "Sur une classe remarquable de courbes et de surfaces algébriques").

If the trinodal quartic

$$y^2z^2+z^2x^2+x^2y^2+2xyz\,(ax+by+cz)=0$$

is capable of being written in the form (13), it is easily seen that we must have $c=1+ab$, or either of two similar relations.

9. The investigation in § 6 will include the case of a quartic with a triple point. For the equation (4) will represent such a quartic, if $a+b+c+d=0$, the triple point being $x=y=z=u$. The equation (8) then gives $(\vartheta+1)^2=0$, and is irrelevant, showing that there are no conics which touch the sides of an infinite number of triangles inscribed in a quartic with a triple point.

On the Theory of Screws in Elliptic Space. (Supplementary Note.)

By Arthur Buchheim, M.A.

[*Read November 13th*, 1884.]

At the January Meeting of the Society, I read a paper "On the Theory of Screws in Elliptic Space," which has since appeared in the *Proceedings*. My object was "to show that the *Ausdehnungslehre* supplies all the necessary materials for a calculus of screws in Elliptic Space." When I wrote that paper, I did not see how the same methods could (except in one obvious and unsatisfactory way) be extended to other kinds of space. In a paper on biquaternions, which is to appear in the *American Journal of Mathematics*, I have developed Clifford's calculus, in such a way as to make the methods and formulæ apply simultaneously to the three kinds of uniform space. While writing that paper, I saw how Grassmann's methods may be extended so as to give the metric formulæ for all kinds of space. This extension I explain in the present note.

The fundamental idea of the extension in question is that the symbol K no longer denotes the *Ergänzung* (complement) of a figure, but its polar with respect to the absolute. Besides this extension of the *Ausdehnungslehre*, I give a few kinematical investigations.

1.

In this note I use Professor Cayley's matrix notation, especially as applied to quadrics. Judging from the look of some recent work on Theta Functions, this notation does not seem to be as well known as it should be; I accordingly begin by explaining it.

Consider any symmetrical (or, as I prefer to call it, self-conjugate) matrix, and the substitution defined by it. To fix the ideas, suppose we have four variables; we take

$$(\xi_1, \xi_2, \xi_3, \xi_4) = \left(\begin{array}{cccc} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & d \end{array}\right)\!\!\between x_1, x_2, x_3, x_4). \quad (*)$$

Now, using ξ to denote the set $(\xi_1, \xi_2, \xi_3, \xi_4)$, and, in the same way, using x to denote the set (x_1, x_2, x_3, x_4), and using A to denote the matrix of the substitution, I write this equation

$$\xi = Ax,$$

viz., this is a symbolic equation, to be understood as standing for the developed equation (*) above.

Now, let y be a new set: that is, we write y for (y_1, y_2, y_3, y_4), and

define $$xy = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4.$$

Then we get $$x\xi = (abcdfghlmn \between x_1, x_2, x_3, x_4)^2.$$

But we had $$\xi = Ax.$$

Therefore, writing Ax^2 for $(Ax)\,x$, we can say that, if A denotes the matrix

$$\left(\begin{array}{cccc} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & d \end{array}\right)$$

then we can use Ax^2 to denote the quadric

$$(abcdfghlmn \between x_1, x_2, x_3, x_4)^2.$$

I call A the matrix of the quadric, and shall refer to the quadric as the quadric A.

It is hardly necessary to point out that, if x is a point, ξ is its polar plane with respect to the quadric A, and that the "tangential" equation is $A^{-1}\xi^2 = 0$, or that what precedes applies to sets containing any number of letters.

2.

As before, I denote the points of reference by e_1, e_2, e_3, e_4, and the complement (*Ergänzung*) by K. I call to mind that, if the four coordinates of a plane are ξ_1, ξ_2, ξ_3, ξ_4, the plane itself is

$$\xi = \xi_1 Ke_1 + \xi_2 Ke_2 + \xi_3 Ke_3 + \xi_4 Ke_4.$$

Now consider the polar plane of e_1 with respect to the quadric A of (1); its coordinates are $(ahgl)$: that is, the plane is

$$aKe_1 + hKe_2 + gKe_3 + lKe_4;$$

and we get similar expressions for the polar planes of the three other vertices of the tetrahedron of reference.

We shall take the quadric A as absolute, and we denote the polar planes of e_1, &c. by ωe_1, &c. We have, by what precedes,

$$(\omega e_1, \omega e_2, \omega e_3, \omega e_4) = \left(\begin{array}{|cccc|} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & d \end{array} \between Ke_1, Ke_2, Ke_3, Ke_4\right);$$

say, this is

$$(\omega e_1, \omega e_2, \omega e_3, \omega e_4) = (A \between Ke_1, Ke_2, Ke_3, Ke_4),$$

and then, if we define that

$$\omega \Sigma x_i e_i = \Sigma x_i (\omega e_i),$$

ω is obviously an operator which changes any point into its polar.

In exactly the same way, if $\omega e_2 e_3 e_4$ denotes the pole of $e_2 e_3 e_4$, and $\omega e_2 e_3$ denotes the polar of $e_2 e_3$, we get

$$(\omega e_2 e_3 e_4, \omega e_3 e_1 e_4, \omega e_1 e_2 e_4, \omega e_3 e_2 e_1) = (A' \between e_1, e_2, e_3, e_4),$$

$$(\omega e_2 e_3, \omega e_3 e_1, \omega e_1 e_2, \omega e_1 e_4, \omega e_2 e_4, \omega e_3 e_4) = (A'' \between e_1 e_4, e_2 e_4, e_3 e_4, e_2 e_3, e_3 e_1, e_1 e_2),$$

where A', A'' are self-conjugate matrices of the fourth and sixth sides respectively, and are, in fact, the matrices of the plane and line equations of the absolute.

If A is the matrix unity, A', A'' are also the units of their own orders, and each equation of the absolute is of the form $\Sigma x^2 = 0$, and then we have simply $\omega = K$; and for any other form of the absolute ω has the same meaning as K has for the special form.

Now, though the absolute was not explicitly used there, it is tolerably obvious that all the formulæ of the paper on the Theory of Screws in Elliptic Space have reference to this special form of the equation

the absolute. To see this, we have only to consider the expression for the distance between two points; this is given by

$$\cos(xy) = \frac{xKy}{\sqrt{xKx}\sqrt{yKy}}$$

$$= \frac{x_1y_1+x_2y_2+x_3y_3+x_4y_4}{(x_1^2+x_2^2+x_3^2+x_4^2)^{\frac{1}{2}}\,(y_1^2+y_2^2+y_3^2+y_4^2)^{\frac{1}{2}}}.$$

But this is what the ordinary expression for the distance becomes if we take $x_1^2+x_2^2+x_3^2+x_4^2=0$ as the equation of the absolute.

It is now obvious that the formulæ of the paper become applicable to any form of the equation of the absolute, provided that for K, denoting the complement, we substitute ω, denoting the polar with respect to the absolute.

I use ω partly because it is necessary to have two symbols for the two things, and partly to make the formulæ look like the biquaternion formulæ of the paper above referred to. In the following section, I take a special form of the equation of the absolute, and show how we get formulæ applicable to the three kinds of space.

3.

I shall now, to my own and my readers' relief, discard suffixes, and use the ordinary notations $(xyzw)$ for the coordinates of a point, and $(lmnp)$, $(abcfgh)$ for planes and screws respectively. I also use α, β, γ, δ for e_1, e_2, e_3, e_4.

Let the point equation of the absolute be

$$e^2(x^2+y^2+z^2)+w^2=0.$$

Then the plane equation will be

$$l^2+m^2+n^2+e^2p^2=0,$$

and the line equation will be

$$f^2+g^2+h^2+e^2(a^2+b^2+c^2)=0.$$

It is obvious that $e^2=\pm 1$ gives elliptic and hyperbolic space. It is not quite so obvious that $e^2=0$ gives parabolic space, if $w=0$ is the plane infinity; but it is not hard to see if we remember,—

(1) That the absolute is a curve, viz., the "circle at infinity" taken twice over, so that its point equation is (plane infinity)$^2=0$;

(2) That therefore a line touches the absolute if it cuts the "circle at infinity"; and that

(3) f, g, h are proportional to the direction cosines of $(abcfgh)$.

We have

$$(\omega a,\ \omega\beta,\ \omega\gamma,\ \omega\delta) = \left(\begin{array}{cccc} e^2, & 0, & 0, & 0 \\ 0, & e^2, & 0, & 0 \\ 0, & 0, & e^2, & 0 \\ 0, & 0, & 0, & 1 \end{array}\right)\!\!\between\, \beta\gamma\delta,\ \gamma a\delta,\ a\beta\delta,\ \gamma\beta a),$$

$$(\omega\beta\gamma\delta,\ \omega\gamma a\delta,\ \omega a\beta\delta,\ \omega\gamma\beta a) = \left(\begin{array}{cccc} 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & e^2 \end{array}\right)\!\!\between\, a,\ \beta,\ \gamma,\ \delta),$$

$$(\omega\beta\gamma,\ \omega\gamma a,\ \omega a\beta,\ \omega a\delta,\ \omega\beta\delta,\ \omega\gamma\delta) = \left(\begin{array}{cccccc} 0, & 0, & 0, & e^2, & 0, & 0 \\ 0, & 0, & 0, & 0, & e^2, & 0 \\ 0, & 0, & 0, & 0, & 0, & e^2 \\ 1, & 0, & 0, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0, & 0, & 0 \\ 0, & 0, & 1, & 0, & 0, & 0 \end{array}\right)\!\!\between\, \beta\gamma,\ \gamma a,\ a\beta,\ a\delta,\ \beta\delta,\ \gamma\delta).$$

It is necessary to change the definition of $[ab]$ (p. 91) and to write

$$e^{-1}\sin e\,[ab] = \frac{T(ab)}{Ta\,Tb} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\theta),$$

where, of course, $\qquad Ta = +\sqrt{a\,.\,\omega a}.$

To show how the formulæ apply to parabolic space, I take the expression for the distance between two points and the expression for the axis of a screw.

We take $a = (xyzw)$, $b = (x'y'z'w')$; and we must remember that, in parabolic space, $w = 0$ is the plane infinity, so that $w = \text{const.}$ for all points not at infinity, and we can take $w = 1$.

We have

$$a = xa + y\beta + z\gamma + w\delta,$$

$$\omega a = e^2 x\beta\gamma\delta + e^2 y\gamma a\delta + e^2 z a\beta\delta + w\gamma\beta a.$$

Therefore $\qquad T^2 a = a\,\omega a = e^2(x^2+y^2+z^2) + w^2.$

Moreover, if $(abcfgh)$ are the coordinates of the line ab, we get easily

$$T^2(ab) = f^2+g^2+h^2+e^2(a^2+b^2+c^2).$$

Therefore, putting in the values of the coordinates, we get

$$e^{-2}\sin^2 e\,[ab] = \frac{\left[\begin{array}{l}(xw'-x'w)^2+(yw'-y'w)^2+(zw'-z'w)^2 \\ +e^2\{(yz'-y'z)^2+(zx'-z'x)^2+(xy'-x'y)^2\}\end{array}\right]}{\{e^2(x^2+y^2+z^2)+w^2\}\{e^2(x'^2+y'^2+z'^2)+w'^2\}}.$$

Now, if we take $e=0$ and $w=w'=1$, we get

$$[ab]^2 = (x-x')^2+(y-y')^2+(z-z')^2,$$

which is right.

Now take the expression for the axis of a screw: using (θ) above instead of the equation on p. 91, the expression for the axis on p. 94

becomes $$b = a\cos\frac{e\phi}{2} - e^{-1}\,\omega a \sin\frac{e\phi}{2},$$

where $$e^{-1}\sin e\phi = \frac{a^2}{T^2 a}.$$

Now we need only consider the case $e=0$, and then we have

$$b = a - \omega a\,.\,\frac{\phi}{2},$$

$$\phi = \frac{a^2}{T^2 a},$$

or $$b = a - \frac{a^2}{2T^2 a}\,.\,\omega a = a - \lambda\,.\,\omega a \text{ say}.$$

Now let the coordinates of a be $(abcfgh)$: then the coordinates of ωa are $(e^2f,\ e^2g,\ e^2h,\ a, b, c)$, and we get, since $e=0$,

$$b = (a,\ b,\ c,\ f-\lambda a,\ g-\lambda b,\ h-\lambda c).$$

But $$\lambda = \frac{a^2}{2T^2 a} = \frac{2\,(af+bg+ch)}{2\,(f^2+g^2+h^2)}.$$

Now, in a paper "On the application of Quaternions to the Theory of the Linear Complex and the Linear Congruence" (*Mess. of Math.*, Vol. XII., p. 129), I have (allowing for a mistake in sign) given

$$\beta,\quad \alpha - \frac{S\alpha\beta}{\beta^2}\,.\,\beta$$

as the vector coordinates of the axis of the screw (α,β), where $\alpha = fi+gj+hk$, $\beta = ai+bj+ck$: and these values obviously agree with the value of b just given. There are one or two points in which the formulæ of the paper on the Theory of Screws require modification.

The two conditions for parallelism are

$$a \pm e^{-1}\omega a = \lambda\,(b \pm e^{-1}\omega b),$$

and the definition of a vector is

$$a \pm e^{-1}\omega a = 0.$$

Writing
$$\xi = \frac{1+e^{-1}\omega}{2},$$
$$\eta = \frac{1-e^{-1}\omega}{2},$$

I find it convenient to replace Clifford's *right* and *left* by ξ, η used as prefixes: that is to say, a, b are ξ-parallel if $\xi a = \lambda \xi b$, and a is a ξ-vector if $\xi a = 0$: there are, of course, corresponding definitions of η-parallels, and of the η-vector.

It is easy to prove and important to notice, that, if a, b are any two screws, we have

$$\omega a \,.\, \omega b = e^2\,(ab).$$

For the next section the following definition and theorem are required:—If x is any point, and if a is any screw, xa is called the plane corresponding to x with respect to a: if a is a line, xa is the plane joining x, a: in the same way, if x is a plane, we have a point xa which, in the case in which a is a line, becomes the point of intersection of the line and plane.

If $x \equiv (xyzw)$ and $a \equiv (abcfgh)$ and $xa \equiv (lmnp)$, we have

$$(lmnp) = \left(\begin{vmatrix} 0, & h, & -g, & a \\ -h, & 0, & f, & b \\ g, & -f, & 0, & c \\ -a, & -b & -c, & 0 \end{vmatrix} \between xyzw\right).$$

If $(xyzw)$ is on the line $(abcfgh)$, $(lmnp) \equiv 0$.

The plane $(lmnp)$ always passes through $(xyzw)$.

4.

A motion* is defined as a linear transformation not altering the absolute: therefore, if $(xyzw)$ moves to $(x'y'z'w')$, we have

$$(x'y'z'w') = (1+X \between xyzw),$$

where X is a matrix.

I now suppose the motion to be infinitesimal, and for $1+X$ I write

* *Cf.* Lindemann, *Math. Ann.*, Band VII.

$\lambda+\epsilon$, where λ is a scalar matrix, differing infinitesimally from unity, and ϵ is an infinitesimal matrix to be determined. Now let A be the matrix of the absolute: then, since $\lambda+\epsilon$ is to be an automorphic of A, we have, using ϵ' to denote the conjugate of ϵ,

$$(\lambda+\epsilon')\,A\,(\lambda+\epsilon)=\rho A,$$

ρ being a scalar, and $\rho-1$ infinitesimal; multiplying out, and neglecting infinitesimals of the second order, we get

$$\lambda^2A+\epsilon'A+A\epsilon=\rho A.$$

Therefore all the conditions of the problem are satisfied, provided we take

$$\lambda^2=\rho,$$

$$A\epsilon+\epsilon'A=0.$$

Now this last condition asserts that $A\epsilon$ is a skew matrix, say

$$A\epsilon=\eta=\begin{pmatrix} 0 & h & -g & a \\ -h & 0 & f & b \\ g & -f & 0 & c \\ -a & -b & -c & 0 \end{pmatrix}.$$

Then, if $\eta\,(xyzw)$ represents a plane, $\epsilon\,(xyzw)=A^{-1}\eta\,(xyzw)$ will represent the pole of that plane. Now we have

$$(x'y'z'w')=\lambda\,(xyzw)+\epsilon\,(xyzw),$$

that is, the new position of the point $(xyzw)$ is on the line joining the point to the pole of the plane $\eta\,(xyzw)$; but it is obvious, from (3), that this plane is the plane corresponding to $(xyzw)$ with respect to the screw $(abcfgh)$; moreover the line joining any point to the pole of a plane is (by definition) at right angles to the plane. Combining all this, we get the theorem:—Every infinitesimal motion of a rigid body is defined by a certain screw, in such wise that every point of the body moves along the normal to the plane corresponding to the point with respect to the screw, and that every plane of the body turns about the normal to the point corresponding to the plane with respect to the polar screw.

On this I remark:—(1) The second part is inserted in virtue of the principle of duality; (2) The normal to a point, in a plane, is the intersection of the plane with the polar plane of the point; (3) If a is any screw, ωa is the polar screw.

If a is a line, it is obvious that a point moves at right angles to the plane joining it to the line, and that the motion is of the nature of a

rotation about the line: for we have

$$(x'y'z'w') = (\lambda + A^{-1}\eta \between xyzw);$$

but, if $(abcfgh)$ is a line, we have for all points on it $\eta\,(xyzw) \equiv 0$, and therefore $A^{-1}\,\eta\,(xyzw) \equiv 0$, and therefore

$$(x'y'z'w') = (\lambda \between xyzw).$$

Therefore all points on the line are unaltered, that is to say, the motion is a rotation about the line. But it appears, in precisely the same way, that all planes through the polar of the line are unaltered so that the motion is at the same time a translation along the polar. It is a well known and easily proved theorem, that any rotation about a line is at the same time a translation along its polar. For, since the absolute is unaltered, if a point P moves to P', the plane ωP will move to $\omega P'$; and therefore, if all points $\lambda P_1 + \mu P_2$ are unaltered, all planes $\lambda\omega P_1 + \mu\omega P_2$ are unaltered: but the points are the points of a straight line, and the planes are the planes through its polar; moreover, a motion which does not affect the points of a line is a rotation about the line, and a motion which does not affect the planes through a line is a translation along the line. The theorem is therefore proved.

It is worth while to consider space of more than three dimensions. It is obvious that all the investigations of this section apply, and we get the theorems:—

"Every infinitesimal motion of a rigid body in a space of n dimensions is defined by a certain form a of order $n-2$ in the units of reference, in such wise that any point x moves along the normal to the $(n-1)$-flat xa."

"Every rotation about an r-flat is at the same time a rotation about the polar $(n-r-1)$-flat."

5.

I now take the equation of the absolute in the canonical form

$$e^2(x^2+y^2+z^2)+w^2 = 0,$$

and I take $(abcfgh)$ as the coordinates of the screw defining the motion.

Then, if $(xyzw)$ moves to $(x'y'z'w')$, we have

$$(x'y'z'w') = \left(\begin{vmatrix} \lambda & h & -g & a \\ -h & \lambda & f & b \\ g & -f & \lambda & c \\ -e^2a & -e^2b & -e^2c & \lambda \end{vmatrix} \between xyzw\right);$$

and then, if (A, B, C, F, G, H) becomes (A', B', C', F', G', H'), we have

$$(A'B'C'F'G'H') = \left(\begin{array}{cccccc} \lambda^2 & h & -g & 0 & c & -b \\ -h & \lambda^2 & f & -c & 0 & a \\ g & -f & \lambda^2 & b & -a & 0 \\ 0 & e^2c & -e^2b & \lambda^2 & h & -g \\ -e^2c & 0 & e^2a & -h & \lambda^2 & f \\ e^2b & -e^2a & 0 & g & -f & \lambda^2 \end{array}\right)(ABCFGH)$$

we can verify at once that the two screws $(abcfgh)$, $(e^2f, e^2g, e^2h, a, b, c)$ are transformed into themselves.

Moreover, it is a known property, which can be easily verified in the present instance, that if a transformation of line-coordinates is derived, as this is, from a transformation of point-coordinates, then $AF+BG+CH$, and *therefore* $AF'+A'F+BG'+B'G+CH'+C'H$ are invariants; therefore, if we call the screw $(abcfgh)$, A, and say that two screws x, y are reciprocal (in involution) if xy vanishes, we get as the first result that every screw reciprocal to A or to ωA is transformed into a screw reciprocal to A or to wA.

Moreover, since A is absolutely unaltered by the motion, the axes of A are also unaffected by it, and therefore any point on an axis (plane through an axis) is transformed into a point on the same axis (plane through the same axis): therefore every infinitesimal motion is a rotation about an axis of the screw defining the motion, and therefore also a translation along an axis of the screw.

We get also

$$(A' \pm e^{-1}F', B' \pm e^{-1}G', C' \pm e^{-1}H')$$

$$= \left(\begin{array}{ccc} \lambda^2 & \pm e(c \pm e^{-1}h) & \mp e(b \pm e^{-1}g) \\ \mp e(c \pm e^{-1}h) & \lambda^2 & \pm e(a \pm e^{-1}f) \\ \pm e(b \pm e^{-1}g) & \mp e(a \pm e^{-1}f) & \lambda^2 \end{array}\right)(A \pm e^{-1}F, B \pm e^{-1}G, C \pm e^{-1}H).$$

Therefore

(1) If $A \pm e^{-1}F = B \pm e^{-1}G = C \pm e^{-1}H = 0$, $A' \pm e^{-1}F'$, $B' \pm e^{-1}G'$, $C' \pm e^{-1}H'$, all vanish: therefore ξ-vectors are transformed into ξ-vectors; η-vectors into η-vectors.

(2) If
$$\frac{A \pm e^{-1}F}{A_1 \pm e^{-1}F_1} = \frac{B \pm e^{-1}G}{B_1 \pm e^{-1}G_1} = \frac{C \pm e^{-1}H}{C_1 \pm e^{-1}H_1},$$
we have also
$$\frac{A' \pm e^{-1}F'}{A_1' \pm e^{-1}F_1'} = \frac{B' \pm e^{-1}G'}{B_1' \pm e^{-1}G_1'} = \frac{C' \pm e^{-1}H'}{C_1' \pm e^{-1}H_1'};$$

that is, parallel lines or screws become parallel lines or screws.

(3) If $a \pm e^{-1}f = b \pm e^{-1}g = c \pm e^{-1}h = 0$,

we have $$\frac{A' \pm e^{-1}F'}{A \pm e^{-1}F} = \frac{B' \pm e^{-1}G'}{B \pm e^{-1}G} = \frac{C' \pm e^{-1}H'}{C \pm e^{-1}H} = \lambda^2.$$

Therefore, if the screw defining the motion is a vector, every screw becomes a parallel screw.

I now find the distance through which a point moves. I use quaternions to shorten the calculation.

We have, for the distance between the points,

$$e^{-2}\sin^2 e\,[PP'] = \frac{\left[\begin{array}{c}(xw'-x'w)^2+(yw'-y'w)^2+(zw'-z'w)^2 \\ +e^2\{(yz'-y'z)^2+(zx'-z'x)^2+(xy'-x'y)^2\}\end{array}\right]}{\{e^2(x^2+y^2+z^2)+w^2\}\{e^2(x'^2+y'^2+z'^2)+w'^2\}}.$$

If $\rho = xi+yj+zk$, $\rho' = x'i+y'j+z'k$, this is obviously

$$\frac{e^2T^2V\rho\rho' + T^2(w\rho'-w'\rho)}{(e^2T^2\rho+w^2)(e^2T^2\rho'+w'^2)}.$$

Now, in the present case, we have

$$(x'y'z'w') = (\lambda x+hy-gz+aw,\ \lambda y+fz-hx+bw,\ \lambda z+gx-fy+cw,$$
$$\lambda z - e^2ax - e^2by - e^2cz).$$

Now take $\alpha = fi+gj+hk$, $\alpha' = ai+bj+ck$: then we have

$$\rho' = \lambda\rho + V\rho\alpha + w\alpha',$$
$$w' = \lambda w + e^2S\rho\alpha'.$$

We have therefore, if we omit terms which obviously cancel, to calculate

$$X = e^2T^2V\rho\,(V\rho\alpha + w\alpha') + T^2(e^2\rho S\rho\alpha' - wV\rho\alpha - w^2\alpha').$$

Now the first term is

$$e^2T^2\rho T^2(V\rho\alpha + w\alpha') - e^2S^2\rho\,(V\rho\alpha + w\alpha')$$
$$= e^2T^2\rho\,(T^2V\rho\alpha - 2wS\rho\alpha\alpha' + w^2T^2\alpha') - e^2w^2S^2\rho\alpha'.$$

The second term is

$$e^4T^2\rho S^2\rho\alpha' + 2e^2w^2S^2\rho\alpha' + w^2T^2V\rho\alpha - 2w^3S\rho\alpha\alpha' + w^4T^2\alpha'.$$

This gives

$$X = (e^2T^2\rho + w^2)(T^2V\rho\alpha + w^2T^2\alpha' - 2wS\rho\alpha\alpha' + e^2S^2\rho\alpha')$$
$$= (e^2T^2\rho + w^2)\,\{T^2(V\rho\alpha + w\alpha') + e^2S^2\rho\alpha'\}.$$

Moreover, neglecting terms of the second order,

$$e^2T^2\rho' + w'^2 = \lambda^2 (e^2T^2\rho + w^2).$$

Therefore $$e^{-2}\sin^2 ePP' = \frac{T^2 (V\rho\alpha + w\alpha') + e^2S^2\rho\alpha'}{\lambda^2 (e^2T^2\rho + w^2)} \quad \ldots\ldots\ldots\ldots\ldots (\delta).$$

Now introduce the notation of the rest of this paper, calling the screw of the motion A, and we get

$$e^{-1}\sin ePP' = \frac{T(PA)}{\lambda TP} = \frac{TA}{\lambda} . \sin [PA].$$

If A is a vector, we have $\alpha = \pm e\alpha'$, and we get from (δ)

$$e^{-2}\sin^2 ePP' = \frac{e^2T^2V\rho\alpha' + w^2T^2\alpha' + e^2S^2\rho\alpha'}{\lambda^2 (e^2T^2\rho + w^2)}$$

$$= \frac{(e^2T^2\rho + w^2)\, T^2\alpha'}{\lambda^2 (e^2T^2\rho + w^2)}$$

$$= \frac{T^2\alpha'}{\lambda^2}.$$

Therefore the translations of all points are equal.

Moreover we have

$$\rho w' - \;'w - eV\rho\rho'$$

$$= e^2\rho S\rho\alpha' - wV\rho\alpha - w^2\alpha' - eV\rho V\rho\alpha - ewV\rho\alpha'$$

$$= e^2\rho S\rho\alpha' - wV\rho\alpha + w^2\alpha' + e\rho S\rho\alpha - e\alpha\rho^2 - ewV\rho\alpha'$$

$$= e\rho (S\rho\alpha + eS\rho\alpha') - w (V\rho\alpha + eV\rho\alpha') - (e\alpha\rho^2 + \alpha' w^2).$$

Therefore, if $\alpha + e\alpha' = 0$, we get

$$\rho w' - \rho' w - eV\rho\rho' = -\alpha' (w^2 + e^2T^2\rho^2).$$

That is, the left-hand side is constant to a factor *près*; therefore, if the screw of a motion is a vector, the translations of all points are parallel.

In precisely the same way we can prove that, if the screw of a motion is a vector, the axes of rotation of all planes are parallel, and that the amount of rotation is constant for all planes, and equal to the amount of translation of all points.

Now consider the motion of a line: let the coordinates of the line be $(ABCFGH)$, and let $Fi + Gj + Hk = \varpi$, $Ai + Bj + Ck = \rho$; then, if ϖ', ρ' are the corresponding vectors for the new position of the line, we have

$$\varpi' = e^2V\rho\alpha' + V\varpi\alpha + \lambda^2\varpi,$$

$$\rho' = V\rho\alpha + V\varpi\alpha' + \lambda^2\rho.$$

But $e^2 V\rho\alpha' + V\varpi\alpha$, $V\rho\alpha + V\varpi\alpha'$ are the vector coordinates of a screw which I have elsewhere (in the memoir on Biquaternions, already referred to) called the axis of the cylindroid determined by the screws (ϖ, ρ) and (α, α').

Therefore we can say that the new position of a line A is in the cylindroid containing A and the axis of the cylindroid $(A\alpha)$ if α is the screw defining the motion.

In conclusion, I prove Professor Ball's theorem that every ξ-vector is reciprocal to every η-vector.

Let a be a ξ-vector, b an η-vector: then we have

$$\omega a = -ea,$$

$$\omega b = eb.$$

Therefore $$(\omega a)(\omega b) = -e^2 ab.$$

But we have universally

$$(\omega a)(\omega b) = e^2 ab.$$

Therefore, unless e vanishes,

$$ab = -ab,$$

or $$ab = 0.$$

On the Motion of a Viscous Fluid contained in a Spherical Vessel.

By HORACE LAMB, M.A., F.R.S.

[*Read November* 13*th*, 1884.]

In several of the most important problems in Viscosity which have as yet been solved, the fluid is supposed limited, whether externally or internally, by a single spherical, or nearly spherical, boundary. For instance, we have the case of a ball pendulum oscillating in an unlimited mass of fluid (Stokes), the case of a hollow spherical shell filled with liquid, and oscillating by the torsion of a suspending wire (Helmholtz), and so on.* In a previous communication† to the Society, I have given formulæ by means of which all problems of this

* See Hicks, "Report on Hydrodynamics," *B. A. Rep.*, 1882.

† *Proceedings*, T. xiii., p. 51.

kind can be treated in a uniform manner, and have applied them to the discussion of the oscillations of a viscous spheroid. As a further illustration of the use of these formulæ, I now investigate the decay of the motion of a viscous fluid contained in a rigid spherical envelope which is at rest.

Let us begin with the case of an incompressible fluid. The equations of small motion then are (on the usual assumptions),

$$\left.\begin{aligned} \frac{du}{dt} &= -\frac{1}{\rho}\frac{dp}{dx} + \nu\nabla^2 u, \\ \frac{dv}{dt} &= -\frac{1}{\rho}\frac{dp}{dy} + \nu\nabla^2 v, \\ \frac{dw}{dt} &= -\frac{1}{\rho}\frac{dp}{dz} + \nu\nabla^2 w \end{aligned}\right\} \quad \text{.......................(1)},$$

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0 \quad \text{.......................(2)},$$

where u, v, w are the component velocities, ρ is the density, p the pressure, ν the "kinematic viscosity," and ∇^2 stands for

$$d^2/dx^2 + d^2/dy^2 + d^2/dz^2.$$

If we assume that u, v, w, p all vary as $e^{\alpha t}$, the equations (1) may be written

$$\left.\begin{aligned} (\nabla^2 + k^2)\, u &= k^2 \frac{d\phi}{dx} \\ (\nabla^2 + k^2)\, v &= k^2 \frac{d\phi}{dy} \\ (\nabla^2 + k^2)\, w &= k^2 \frac{k\phi}{dz} \end{aligned}\right\} \quad \text{..........................(3)},$$

where $$k^2 = -\alpha/\nu \quad \text{..................................(4)},$$

and $$\phi = -p/\alpha\rho \quad \text{..................................(5)}.$$

From (2) and (3) we deduce

$$\nabla^2\phi = 0 \quad \text{..................................(6)}.$$

Let the origin of coordinates be taken at the centre of the spherical cavity. The solutions of the equations (2) and (3) will be of two distinct types. In the *First Type* we shall have

$$\left.\begin{aligned} u &= \psi_n(kr)\left(y\frac{d}{dz} - z\frac{d}{dy}\right)\chi_n \\ v &= \psi_n(kr)\left(z\frac{d}{dx} - x\frac{d}{dz}\right)\chi_n \\ w &= \psi_n(kr)\left(x\frac{d}{dy} - y\frac{d}{dx}\right)\chi_n \end{aligned}\right\} \quad \text{..................(7)},*$$

* [Cf. *Proceedings*, T. xiii., p. 57.]

where χ_n is a solid harmonic of positive integral degree n, and

$$\psi_n(kr) = 1 - \frac{k^2r^2}{2 \,.\, 2n+3} + \frac{k^4r^4}{2 \,.\, 4 \,.\, 2n+3 \,.\, 2n+5} - \&c. \quad \ldots\ldots(8).$$

If we assume that there is no slipping of the fluid in contact with the vessel, we shall have

$$u = 0, \quad v = 0, \quad w = 0,$$

when $r = a$, the radius of the vessel. This requires

$$\psi_n(ka) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

The roots of this are all real, and the corresponding values of the modulus (τ) of decay are then given by

$$\tau = -\alpha^{-1} = \frac{a^2}{\nu}(ka)^{-2} \text{ *} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

For instance, the lowest root of (9), in the case $n = 1$, is $ka = 4{\cdot}493$,

whence $$\tau = {\cdot}0495 \frac{a^2}{\nu} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

In the case of water, we have $\nu = {\cdot}014$ C. G. S., and

$$\tau = 3{\cdot}5a^2 \text{ seconds},$$

if a be expressed in centimetres.

In the solutions of the *Second Type*, we have

$$\phi = \phi_n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12),$$

$$\left.\begin{aligned}
u &= \frac{d\phi_n}{dx} + (n+1)\,\psi_{n-1}(kr)\frac{d\omega_n}{dx} - n\,\frac{k^2r^{2n+3}}{2n+1 \,.\, 2n+3}\,\psi_{n+1}(kr)\frac{d}{dx}\,\omega_n r^{-2n-1} \\
v &= \frac{d\phi_n}{dy} + (n+1)\,\psi_{n-1}(kr)\frac{d\omega_n}{dy} - n\,\frac{k^2r^{2n+3}}{2n+1 \,.\, 2n+3}\,\psi_{n+1}(kr)\frac{d}{dy}\,\omega_n r^{-2n-1} \\
w &= \frac{d\phi_n}{dz} + (n+1)\,\psi_{n-1}(kr)\frac{d\omega_n}{dz} - n\,\frac{k^2r^{2n+3}}{2n+1 \,.\, 2n+3}\,\psi_{n+1}(kr)\frac{d}{dz}\,\omega_n r^{-2n-1}
\end{aligned}\right\} \ldots\ldots\ldots\ldots(13),$$

where ϕ_n, ω_n are solid harmonics of positive integral degree n. If, as before, we assume that there is no slipping at the boundary, we must

have* $$\phi_n + (n+1)\,\psi_{n-1}(ka)\,\omega_n = 0 \ldots\ldots\ldots\ldots\ldots(14),$$

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* See *Proceedings*, T. XIII., pp. 192, 193.

when $r = a$, and $\psi_{n+1}(ka) = 0$ ....(15).

When the values of ka have been found from this equation, the moduli of the corresponding modes of decay are given as before by (10). When $n = 1$, the lowest root of (15) is $ka = 5{\cdot}764$, which gives

$$\tau = {\cdot}0301\,\frac{a^2}{\nu}.$$

When the harmonics ϕ_n, ω_n are zonal, the character of the motion is most simply expressed by means of a stream-function Ψ; viz., if we take the axis of symmetry as axis of x, and write

$$\varpi = \sqrt{(y^2+z^2)}, \quad \mathfrak{v} = (yv+zw)/\varpi,$$

we have

$$u = \frac{1}{\varpi}\frac{d\Psi}{d\varpi}, \quad \mathfrak{v} = -\frac{1}{\varpi}\frac{d\Psi}{dx} \quad \text{.....................(16)},$$

where

$$\Psi = -n\left(x\omega_n - r^2\frac{d\omega_n}{dx}\right)\{\psi_n(kr)-\psi_n(ka)\}$$

$$= -r\sin\theta\,.\,\frac{d\omega_n}{d\theta}\{\psi_n(kr)-\psi_n(ka)\} \quad \text{...............(17),*}$$

θ denoting the co-latitude, *i.e.*, $\varpi = r\sin\theta$. Thus, if $n = 1$, writing $\omega_1 = Cr\cos\theta$, we have

$$\Psi = Cr^3\sin^2\theta\,\{\psi_1(kr)-\psi_1(ka)\} \quad \text{..............(18).}$$

[It may be worth while, for the sake of comparison, to give the corresponding results for the motion of a viscous liquid in a right circular cylinder. Treating the problem as one of two dimensions, and taking the origin of polar coordinates r, θ in the axis of the cylinder, the component velocities along and perpendicular to the radius vector will be

$$\frac{d\Psi}{rd\theta} \text{ and } -\frac{d\Psi}{dr},$$

respectively, where

$$\Psi = \Sigma\Sigma\left\{\frac{J_n(kr)}{J_n(ka)} - \frac{r^n}{a^n}\right\}\{A_n\cos n\theta + B_n\sin n\theta\},$$

the admissible values of k being determined by

$$kaJ_n'(ka) - nJ_n(ka) = 0,$$

i.e.,

$$J_{n+1}(ka) = 0.$$

* *Proceedings*, T. xiii., p. 204.

Here J_n denotes the Bessel's function of integral order n, and a is the radius of the cylinder. In the case of symmetry about the axis,* we have $n = 0$. The lowest root of $J_1(ka) = 0$ is $ka = 3{\cdot}832$, which gives

$$\tau = {\cdot}0681\,\frac{a^2}{\nu}.$$

If, as before, we put for water $\nu = {\cdot}014$ C. G. S., we find

$$\tau = 4{\cdot}9a^2.\dagger]$$

In combining the foregoing solutions so as to represent the decay of any arbitrary initial motion, we make use of the following principle.‡ If u, v, w be three functions subject to the condition (2), and if

$$2\xi = \frac{dw}{dy} - \frac{dv}{dz},\quad 2\eta = \frac{du}{dz} - \frac{dw}{dx},\quad 2\zeta = \frac{dv}{dx} - \frac{du}{dy} \qquad (19),$$

then the values of u, v, w are determinate throughout any spherical region having its centre at the origin, when we know the values of $xu+yv+zw$ and $x\xi+y\eta+z\zeta$ throughout that region. For, regarding u, v, w as the component velocities of a fluid, if there be two distinct motions satisfying the prescribed conditions, then, in the motion which is the difference of these, we have

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0 \qquad (20),$$

$$xu + yv + zw = 0 \qquad (21),$$

$$x\xi + y\eta + z\zeta = 0 \qquad (22).$$

From (20) and (21), it appears that the lines of flow are closed curves lying on a system of concentric spherical surfaces. Hence the "circulation" round any such line has a finite value. On the other hand, (22) shows that the circulation round any circuit drawn on one of the above spherical surfaces is zero. These conclusions are irreconcilable unless u, v, w are all zero.

The following proof is of a more analytical character. Let us employ the ordinary spherical polar coordinates r, θ, ϕ. The com-

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* Discussed by Stearn, *Quar. Jour. Math.*, T. xvii., p. 90.

† In comparing our numerical results with observation, it should be borne in mind that the theory of viscosity on which the equations (1) are based is of a purely empirical character, and that it probably ceases to represent the facts accurately when the rates of strain du/dx, &c. exceed certain limits. (See B. Elie, *Jour. de Physique*, 1882, p. 224.)

‡ Stated, without proof, *Phil. Trans.*, 1883, p. 533.

ponent velocity along the radius vector is zero, by (21). Let the components along and at right angles to the meridian be Θ, Φ, respectively. Forming the equation of continuity for an element $r \sin\theta\, d\phi\,.\, r\, d\theta\,.\, dr$, we find

$$\frac{d}{d\theta}(\Theta \sin\theta) + \frac{d\Phi}{d\phi} = 0 \quad \text{.......................(23).}$$

This replaces (20). Again, calculating the circulation round an elementary rectangle $r \sin\theta\, d\phi\,.\, r\, d\theta$, which must be zero by (22),

$$\frac{d\Theta}{d\phi} - \frac{d}{d\theta}(\Phi \sin\theta) = 0 \quad \text{.......................(24).}$$

We infer from this that $\Theta\, d\theta + \sin\theta\, \Phi\, d\phi$ is a perfect differential with respect to θ and ϕ, say it $= dF$. The equation (23) then becomes

$$\sin\theta \frac{d}{d\theta}\left(\sin\theta \frac{dF}{d\theta}\right) + \frac{d^2F}{d\phi^2} = 0 \text{.....................(25),}$$

or, writing $\vartheta = \log\tan\frac{\theta}{2}$ (Mercator's Projection),

$$\frac{d^2F}{d\vartheta^2} + \frac{d^2F}{d\phi^2} = 0 \quad \text{..........................(26).*}$$

Since F is necessarily periodic with respect to ϕ, we may assume

$$F = \Sigma\,(R_m \cos m\phi + S_m \sin m\phi).$$

Substituting in (26), and integrating, we have

$$R_m = Ae^{m\vartheta} + Be^{-m\vartheta},$$

which cannot be finite at both limits $\vartheta = \pm\infty$ unless it is everywhere zero. For the same reason $S_m = 0$.

The theorem just proved is useful in various physical problems; but for our present purpose the following corollary is more convenient. The component velocities u, v, w of an incompressible fluid occupying a spherical region having its centre at the origin are determinate, save as to three additive functions of the forms $d\phi/dx$, $d\phi/dy$, $d\phi/dz$, respectively, where $\nabla^2\phi = 0$, when we know the values of $x\xi + y\eta + z\zeta$ and $\nabla^2(xu + yv + zw)$ throughout that region. For, if there be two

* This is a particular case of a general result obtained by Kirchhoff (*Berliner Monatsb.*, 1875, or *Collected Papers*, p. 56), relating to problems of electric conduction in two dimensions, and their transformation by orthomorphic projection.

distinct motions satisfying the prescribed conditions, then in the motion which is the difference of these we have

$$x\xi+y\eta+z\zeta=0,$$

and $$\nabla^2(xu+yv+zw)=0.$$

The latter of these may be written

$$x\left(\frac{d\zeta}{dy}-\frac{d\eta}{dz}\right)+y\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)+z\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)=0.$$

Hence the argument is exactly the same as before, except that ξ, η, ζ are written for u, v, w respectively. We infer that ξ, η, ζ are all zero,

i.e., $$u=\frac{d\phi}{dx},\quad v=\frac{d\phi}{dy},\quad w=\frac{d\phi}{dz}\quad\text{..................(27)},$$

where $$\nabla^2\phi=0.$$

To apply this to the problem in hand, we remark that in the solutions of the First Type we have

$$\left.\begin{aligned}2\xi&=-(n+1)\,\psi_{n-1}(kr)\frac{d\chi_n}{dx}+n\frac{k^2r^{2n+3}}{2n+1\,.\,2n+3}\psi_{n+1}(kr)\frac{d}{dx}\chi_nr^{-2n-1}\\ 2\eta&=\text{\&c.},\quad 2\zeta=\text{\&c.}\end{aligned}\right\}\quad\text{............(28)},$$

and thence $$2(x\xi+y\eta+z\zeta)=-n(n+1)\,\psi_n(kr)\,\chi_n;$$

whilst in those of the Second Type

$$\left.\begin{aligned}2\xi&=-k^2\psi_n(kr)\left(y\frac{d}{dz}-z\frac{d}{dy}\right)\omega_n\\ 2\eta&=\text{\&c.},\quad 2\zeta=\text{\&c.}\end{aligned}\right\}\quad\text{...............(29)},$$

and thence $$x\xi+y\eta+z\zeta=0.$$

Again, in the solutions of the First Type we have

$$xu+yv+zw=0,$$

and in those of the Second Type

$$xu+yv+zw=n\phi_n+n(n+1)\,\psi_n(kr)\,\omega_n,$$

whence $$\nabla^2(xu+yv+zw)=-n(n+1)\,k^2\psi_n(kr)\,\omega_n.$$

Hence we have to determine the harmonics χ_n, ω_n, so that, when $t=0$,

$$\Sigma\Sigma n(n+1)\,\psi_n(kr)\,\chi_n=-2(x\xi_0+y\eta_0+z\zeta_0)\text{.........(30)},$$

and $$\Sigma\Sigma n(n+1)\,k^2\psi_n(kr)\,\omega_n = -\nabla^2(xu_0+yv_0+zw_0) \quad\ldots\ldots(31),$$

where u_0, v_0, w_0 denote the given initial values of u, v, w; and ξ_0, η_0, ζ_0 are supposed derived from these by (19). The summations must include all integral values of n, and all admissible values of k, the latter being given in (30) by $\psi_n(ka) = 0$, and in (31) by $\psi_{n+1}(ka) = 0$. The determination of χ_n, ω_n, so as to satisfy (30) and (31) for all values of r from 0 to a, is effected in the usual manner, by means of the theorems $$\int_0^a \psi_n(k_pr)\,\psi_n(k_qr)\,r^{2n+2}\,dr = 0 \quad\ldots\ldots(32),$$

if k_p, k_q be two different roots of $\psi_n(ka) = 0$ or of $\psi_{n+1}(ka) = 0$, and

$$\int_0^r \{\psi_n(kr)\}^2\,r^{2n+2}\,dr$$
$$= \frac{r^{2n+3}}{2}\left[\{\psi_n(kr)\}^2 + \frac{2n+1}{kr}\,\psi_n(kr)\,\psi_n'(kr) + \{\psi_n'(kr)\}^2\right] \ldots\ldots(33).$$

The latter formula gives

$$\int_0^a \{\psi_n(kr)\}^2\,r^{2n+2}\,dr = \frac{a^{2n+3}}{2}\{\psi_n'(ka)\}^2 \quad\ldots\ldots(34),$$

if $\psi_n(ka) = 0$, and

$$\int_0^a \{\psi_n(kr)\}^2\,r^{2n+2}\,dr = \frac{a^{2n+3}}{2}\{\psi_n(ka)\}^2 \quad\ldots\ldots(35),$$

if $\psi_{n+1}(ka) = 0$. When the ω_n have been found, the corresponding values of ϕ_n are given by (14). We have seen that the values of u, v, w which are obtained in this way may, for $t = 0$, differ from u_0, v_0, w_0 respectively by terms of the form (27). It is easily seen, however, that these terms represent a constituent of the original motion which is destroyed impulsively when the boundary of the fluid (which may previously have been changing its shape or position) is brought to rest in the spherical form.

As an example of the above method, let us suppose that previously to the time $t = 0$ the vessel and its contents were rotating as a whole about the axis of z with uniform angular velocity Ω. We then have $\omega_n = 0$, and also $\chi_n = 0$, except for $n = 1$. The value of χ_1 corresponding to the p^{th} root of

$$\psi_1(ka) = 0 \quad\ldots\ldots(36),$$

will be of the form $C_pz\,.\,e^{-\nu k_p^2t}$. The constants C_p are to be found from the condition $$\Sigma\psi_1(k_pr)\,C_p = -\Omega \quad\ldots\ldots(37).$$

Making use of (32), (34), and the formula

$$\int_0^r \psi_n(kr)\, r^{2n+2}\, dr = \frac{r^{2n+3}}{2n+3}\, \psi_{n+1}(kr) \qquad (38),$$

we obtain, after some transformations,

$$C_p = -\frac{10\Omega}{k_p^2 a^3 \,.\, \psi_2(k_p a)} \qquad (39).$$

The motion of the fluid is then given by

$$\left.\begin{aligned} u &= \quad y\,.\,\Sigma\, C_p\,.\,\psi_1(k_p r)\, e^{-\nu k_p^2 t} \\ v &= -x\,.\,\Sigma\, C_p\,.\,\psi_1(k_p r)\, e^{-\nu k_p^2 t} \\ w &= 0 \end{aligned}\right\} \qquad (40).$$

The total angular momentum of the fluid about the axis of z is

$$\rho \iiint (xv - yu)\, dx\, dy\, dz = -\tfrac{8}{3}\pi\rho\,.\,\Sigma\, C_p e^{-\nu k_p^2 t} \int_0^a r^4 \psi_1(k_p r)\, dr$$

$$= \tfrac{16}{3}\pi\rho a^5 \Omega\,.\,\Sigma\, \frac{e^{-\nu k_p^2 t}}{k_p^2 a^2} \qquad (41),$$

by (38) and (39). Since $\Sigma\,(k_p a)^{-2} = \frac{1}{10}$, this gives, for $t = 0$, $\frac{8}{15}\pi\rho a^5\Omega$, which is right. The couple tending to drag the vessel round, in the direction of its previous rotation, is found by performing $-d/dt$ on (41), viz., it is

$$\tfrac{16}{3}\pi\rho\nu a^3\,.\,\Sigma\, e^{-\nu k_p^2 t} \qquad (42).$$

In like manner, the kinetic energy of the fluid is readily found to be

$$= \tfrac{8}{3}\pi\rho a^5 \Omega^2\,.\,\Sigma\, \frac{e^{-2\nu k_p^2 t}}{k_p^2 a^2} \qquad (43).$$

When the fluid filling the spherical vessel is gaseous, the equations (1) are replaced by

$$\left.\begin{aligned} \frac{du}{dt} &= -\frac{1}{\rho_0}\frac{dp}{dx} + \nu\nabla^2 u + \frac{\nu}{3}\frac{d\sigma}{dx} \\ \frac{dv}{dt} &= -\frac{1}{\rho_0}\frac{dp}{dy} + \nu\nabla^2 v + \frac{\nu}{3}\frac{d\sigma}{dy} \\ \frac{dw}{dt} &= -\frac{1}{\rho_0}\frac{dp}{dz} + \nu\nabla^2 w + \frac{\nu}{3}\frac{d\sigma}{dz} \end{aligned}\right\} \qquad (44),$$

$$\frac{ds}{dt} + \sigma = 0 \qquad (45).$$

Here $$\sigma = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \quad\text{.........................(46)},$$

$$s = (\rho - \rho_0)/\rho_0,$$

where ρ_0 is the equilibrium density, and the rest of the notation is the same as before.

If we ignore the effects of conduction and radiation of heat, we have

$$p = p_0 + c^2\rho_0 s \quad\text{.............................(47)},$$

where p_0 is the equilibrium pressure, and c is the velocity of sound in the gas. Hence we obtain from (44) and (45), by differentiation,

$$\frac{d^2s}{dt^2} = c^2\nabla^2 s + \tfrac{4}{3}\nu\nabla^2 \frac{ds}{dt} \quad\text{..........................(48)}.$$

If we now assume that u, v, w, &c. all vary as $e^{\alpha t}$, this becomes

$$(\nabla^2 + h^2)\, s = 0 \quad\text{.............................(49)},$$

provided $$h^2 = -\frac{\alpha^2}{c^2 + \frac{4}{3}\nu\alpha} \quad\text{..........................(50)}.$$

Also the equations (44) may be put in the forms

$$\left.\begin{aligned} (\nabla^2 + k^2)\, u &= (k^2 - h^2)\frac{d\phi}{dx} \\ (\nabla^2 + k^2)\, v &= (k^2 - h^2)\frac{d\phi}{dy} \\ (\nabla^2 + k^2)\, w &= (k^2 - h^2)\frac{d\phi}{dz} \end{aligned}\right\} \quad\text{.....................(51)},$$

where $$\phi = \frac{\alpha}{h^2}\, s = -\frac{1}{h^2}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) \quad\text{..............(52)},$$

provided $$k^2 = -\alpha/\nu \quad\text{................................(53)}.$$

These equations are satisfied by

$$u = \frac{d\phi}{dx},\quad v = \frac{d\phi}{dy},\quad w = \frac{d\phi}{dz},$$

where ϕ is any solution of $(\nabla^2 + h^2)\,\phi = 0$ ....(54).

Hence the complete solution of (51) and (52), subject to the condition

of finiteness at the origin, is

$$u = \frac{d\phi}{dx} + \Sigma\left\{(n+1)\,\psi_{n-1}(kr)\frac{d\omega_n}{dx} - n\frac{k^2r^{2n+3}}{2n+1\,.\,2n+3}\psi_{n+1}(kr)\frac{d}{dx}\omega_n r^{-2n-1}\right\}$$

$$+\Sigma\psi_n(kr)\left(y\frac{d}{dz} - z\frac{d}{dy}\right)\chi_n \;\ldots\ldots\ldots\ldots\ldots\ldots(55),$$

with symmetrical formulæ for v, w. Here ϕ denotes the general solution of (54), viz., $\phi = \Sigma\psi_n(hr)\,\phi_n \ldots\ldots\ldots\ldots\ldots\ldots(56)$,

ϕ_n denoting, like ω_n and χ_n, a solid harmonic of positive integral degree n.

As before, the solutions are of two distinct types. In those of the *First Type*, expressed by the terms in χ_n, there is no variation of density, and the theory is therefore exactly the same as before. For air at ordinary temperatures we may put $\nu = \cdot 15$ C. G. S., about, and the formula (11) for the modulus of decay of the most persistent motion of this type then gives

$$\tau = \cdot 33a^2.$$

In the solutions of the *Second Type*, expressing that $xu+yv+zw$ vanishes for $r = a$, we find

$$\{ha\psi_n'(ha) + n\psi_n(ha)\}\,\phi_n + n\,(n+1)\,\psi_n(ka)\,\omega_n = 0 \;\ldots\ldots(57).$$

Again, if u, v, w severally vanish at the boundary, then since

$$\frac{d}{dx}\psi_n(hr)\,\phi_n = \psi_{n-1}(hr)\frac{d\phi_n}{dx} + \frac{h^2r^{2n+3}}{2n+1\,.\,2n+3}\psi_{n+1}(hr)\frac{d}{dx}\phi_n r^{-2n-1},$$

we infer that we must also have

$$h^2a^2\psi_{n+1}(ha)\,\phi_n - n\,.\,k^2a^2\psi_{n+1}(ka)\,\omega_n = 0 \;\ldots\ldots\ldots\ldots(58).$$

From (57) and (58), we obtain

$$\{ha\psi_n'(ha) + n\psi_n(ha)\}\,k^2a^2\psi_{n+1}(ka) + (n+1)\,h^2a^2\psi_{n+1}(ha)\,\psi_n(ka) = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots(59).$$

This equation determines the admissible values of α.

Now, from (50) and (53), we have

$$h^2a^2 = \frac{\nu\alpha}{c^2 + \frac{4}{3}\nu\alpha}\,k^2a^2 = -\frac{k^4a^4}{\frac{c^2a^2}{\nu^2} - \frac{4}{3}k^2a^2}.$$

For air at 0° C., we have $c = 3\cdot 32\times 10^4$, $\nu = \cdot 15$, C. G. S., so that in all

cases of interest c^2a^2/ν^2 is a large number. The more important solutions of (59) are then readily obtained. In the first place, we may have ka equal to any one of the lower roots of

$$\psi_{n+1}(ka) = 0 \quad\text{.............................}(60),$$

in which case k^2/h^2 is small. The equations (56) and (57) then show that $s = 0$, nearly, so that the corresponding modes of motion differ little from those represented by the solutions of the Second Type in the case of an incompressible fluid. But we may also have ha equal to one of the lower roots of

$$ha\psi_n'(ha) + n\psi_n(ha) = 0 \quad\text{........................}(61),$$

approximately, in which case ka will be large. For, when ka is large,

we have $$\psi_n(ka) = (-)^n . 3 . 5 \ldots 2n+1 . \frac{\sin\left(ka + n\dfrac{\pi}{2}\right)}{(ka)^{n+1}} \quad\text{.........}(62),$$

approximately, so that $\psi_n(ka)/k^2a^2\psi_{n+1}(ka)$ is of the order $1/ka$, and the second term of (59) is small compared with the first. If in (50) we put $\nu = 0$, the equation (61) determines the proper tones of a gas contained in a spherical cavity, when viscosity is neglected.* This subject has been fully discussed by Lord Rayleigh in the Society's *Proceedings*, t. 4, p. 93. To calculate the effect of viscosity on the vibrations, we remark that, from (50), we have $\alpha = ich$, nearly,† and

thence $$k = \sqrt{-\alpha/\nu} = (1-i)\,q,$$

if $$q^2 = \frac{\vartheta c}{2\nu a},$$

ϑ being one of the lower roots of

$$\vartheta\psi_n'(\vartheta) + n\psi_n(\vartheta) = 0 \quad\text{........................}(63).$$

Hence, by (62), $$\frac{\psi_n(ka)}{k^2a^2\psi_{n+1}(ka)} = -\frac{1}{2n+3} \cdot \frac{\tan\left(ka + n\dfrac{\pi}{2}\right)}{ka}$$

$$= -\frac{1}{2n+3} \cdot \frac{1-i}{2qa},$$

* Viz., we then have a velocity-potential ϕ subject to (54), and therefore given by (56). Expressing that $d\phi/dr = 0$ for $r = a$, we obtain (61).

† It will be seen that, except when $n = 0$, the error involved in this assumption is of the second order of small quantities as compared with the quantity ϵ, below, which we are evaluating.

approximately. If then, in (59), we write

$$ha = \vartheta + \epsilon,$$

where ϵ is small, we find, after several reductions,*

$$\epsilon = \frac{1-i}{2qa} \cdot \frac{n(n+1)\vartheta}{n(n+1)-\vartheta^2} \quad \text{.......................(64).}$$

Hence $$\alpha = ich = \frac{ic}{a}(\vartheta + \epsilon),$$

the real part of which is

$$-\frac{n(n+1)\vartheta}{\vartheta^2 - n(n+1)} \cdot \frac{c}{2qa^2}.$$

Substituting for q its value, we find that the time τ in which the vibrations fall to $1/e$ of their original amplitude is given by

$$\tau = \frac{\vartheta^2 - n(n+1)}{n(n+1)\vartheta^4} \cdot \sqrt{\frac{2a^3}{\nu c}} \quad \text{.......................(65).}$$

The foregoing investigation does not apply to the *radial* vibrations. When $n = 0$, the formulæ (55) reduce to their first terms, and we have the *exact* equation $\psi_0'(ha) = 0$ ....(66).

If ϑ be a root of this, we find, from (50),

$$\alpha = i\vartheta \frac{c}{a} - \frac{2}{3}\vartheta^2 \frac{\nu}{a^2},$$

so that the modulus of decay is

$$\tau = \frac{3}{2\vartheta^2} \cdot \frac{a^2}{\nu} \quad \text{..............................(67).}$$

We notice that, omitting numerical factors, the ratio of (67) to (65) is of the order $\sqrt{ac/\nu}$. In all cases to which our approximations are applicable this is a large number, so that the radial vibrations are much more slowly extinguished by viscosity than those corresponding to values of $n > 0$. This is readily accounted for. In the latter modes the condition that there is to be no slipping of the fluid in contact with the vessel implies a relatively greater amount of distortion of the fluid elements, and consequent dissipation of energy, in the superficial layers of the gas.

It appears from the general theory of the vibrations of a system

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* For the formulæ of reduction employed, see *Proceedings*, t. 13, p. 190.

subject to small viscous forces, as it is developed, for instance, in Lord Rayleigh's *Sound*, Chap. v., that the rate of decay of any fundamental mode may be found by equating the rate of decrease of the energy to the dissipation, these latter quantities being calculated as if the nature of the mode and its frequency were unaltered by the viscosity. This method has been applied in the Society's *Proceedings*, t. 13, p. 63, to the oscillations of a liquid spheroid; and it might be used in the present problem to verify the formula (67) for the modulus of decay of the *radial* vibrations. But it is important to remark that the same method is not applicable to the case of $n > 0$, and would, in fact, lead to a result of an altogether different form from (65); viz., it would make τ equal to a^2/ν multiplied by a numerical factor. If, indeed, we assume that there may be slipping at the boundary and introduce a coefficient β of sliding friction, as in § 181 of my "Motion of Fluids," then, provided β/a and ν/a^2 are both small in comparison with the rapidity (α) of the motion, the theory in question is applicable, and the method indicated must lead to a correct result. But, in the foregoing investigations, we have (in effect) assumed β to be infinite, so that we can draw no conclusions from the general theory.

In the gravest radial vibration, we have $\vartheta = 4{\cdot}493$, and (67) then gives

$$\tau = {\cdot}0743\,\frac{a^2}{\nu},$$

or, for $\nu = {\cdot}15$ (air), $\tau = {\cdot}49a^2$. In the gravest mode of the class $n = 1$, we have $\vartheta = 2{\cdot}081$, and thence, by (65),

$$\tau = 1{\cdot}143\sqrt{\frac{a^3}{\nu c}}.$$

Assuming $c = 3{\cdot}32 \times 10^4$, $\nu = {\cdot}15$, this becomes $\tau = {\cdot}016a^{\frac{3}{2}}$. For $n=2$, the lowest root of (63) is $\vartheta = 3{\cdot}342$, which gives

$$\tau = {\cdot}666\sqrt{\frac{a^3}{\nu c}}.$$

There can be no doubt that, as a matter of fact, the damping of the vibrations of air (or other gas) contained in a closed vessel is considerably more rapid than these results would indicate. It has been pointed out by Kirchhoff* that the effect of conduction and radiation of heat on the vibrations of a gas must be of at least the same order of magnitude as that of viscosity. The application of Kirchhoff's equations, in which account is taken of all these influences, to the

* *Pogg. Ann.*, t. 134 (1868); *Collected Papers*, p. 540.

problem of this paper does not appear to present any great difficulty; but, as the results would involve a certain constant whose precise value we have at present no means of estimating, I have not thought it worth while to go through the calculations. It is plain, however, that the general effect of the thermal processes must be equivalent to an increase in the viscosity.*

[The effect of viscosity on the transverse vibrations of a gas contained in a circular cylinder can be calculated in a similar manner. If we take the axis of the cylinder as axis of z, the equations of motion may be written

$$(\nabla_1^2+k^2)\,u = (k^2-h^2)\,\frac{d\phi}{dx},$$

$$(\nabla_1^2+k^2)\,v = (k^2-h^2)\,\frac{d\phi}{dy},$$

$$\frac{du}{dx}+\frac{dv}{dy}+h^2\phi = 0,$$

where

$$\nabla_1^2 = \frac{d^2}{dx^2}+\frac{d^2}{dy^2},$$

and the rest of the notation is as before. The solution of these equations is

$$u = \frac{d\phi}{dx}+\frac{d\psi}{dy},$$

$$v = \frac{d\phi}{dy}-\frac{d\psi}{dx},$$

where

$$(\nabla_1^2+h^2)\,\phi = 0,$$

$$(\nabla_1^2+k^2)\,\psi = 0.$$

If we introduce polar coordinates r, θ in the plane xy, we have

$$xu+yv = r\frac{d\phi}{dr}+\frac{d\psi}{d\theta},$$

$$yu-xv = -\frac{d\phi}{d\theta}+r\frac{d\psi}{dr}.$$

Both these expressions must vanish when $r=a$, the radius of the

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* See the investigation by Stokes, of which an account is given in Rayleigh's *Sound*, § 247.

cylinder. Hence we assume

$$\phi = A_n J_n(hr) \cos n\theta,$$

$$\psi = B_n J_n(kr) \sin n\theta,$$

and obtain
$$A_n \,.\, haJ_n'(ha) + B_n \,.\, nJ_n(ka) = 0,$$

$$A_n \,.\, nJ_n(ha) + B_n \,.\, kaJ_n'(ka) = 0.$$

Eliminating, we obtain the following equation to determine a:

$$\frac{haJ_n'(ha)}{nJ_n(ha)} = \frac{nJ_n(ka)}{kaJ_n'(ka)}.$$

In the solutions which are of most interest, ha is approximately equal to one of the lower roots of $J_n'(ha) = 0$,* and ka is large. Hence

$$J_n(ka) = \text{const.} \times \frac{\cos\left(ka - \frac{\pi}{4} - \frac{n\pi}{2}\right)}{(ka)^{\frac{1}{2}}},$$

approximately, and

$$\frac{nJ_n(ka)}{kaJ_n'(ka)} = -\frac{n}{ka} \cot\left(ka - \frac{\pi}{4} - \frac{n\pi}{2}\right).$$

If, as before, we put
$$k = (1-i)\,q,$$

we find
$$\cot\left(ka - \frac{\pi}{4} - \frac{n\pi}{2}\right) = i,$$

nearly, so that the equation to determine a becomes

$$\frac{haJ_n'(ha)}{nJ_n(ha)} = \frac{n}{2qa}(1-i).$$

Assuming
$$ha = \vartheta + \epsilon,$$

where
$$J_n'(\vartheta) = 0,$$

and ϵ is small, we have

$$haJ_n'(ha) = \vartheta J_n''(\vartheta) \,.\, \epsilon = \frac{n^2 - \vartheta^2}{\vartheta} \,.\, J_n(\vartheta) \,.\, \epsilon,$$

by the differential equation of Bessel's Functions. Hence

$$\epsilon = \frac{n^2\vartheta}{n^2 - \vartheta^2} \cdot \frac{1}{2qa}(1-i).$$

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* Rayleigh's *Sound*, § 339.

Now, with sufficient approximation, we have

$$a = ich = \frac{ic}{a}(\vartheta + \epsilon),$$

the real part of which is $-\dfrac{n^2\vartheta}{\vartheta^2 - n^2} \cdot \dfrac{c}{2qa^2}.$

Since $$q = \sqrt{\frac{\vartheta c}{2\nu a}},$$

the modulus of decay is given by

$$\tau = \frac{\vartheta^2 - n^2}{n^2 \vartheta^{\frac{1}{2}}} \sqrt{\frac{2a^3}{\nu c}},$$

which may be compared with (65).

The above treatment does not apply to the radial vibrations ($n = 0$). In this case, we have $\psi = 0$, and

$$\phi = J_0(hr),$$

with the *exact* equation $J_0'(ha) = 0.$

If ϑ be a root of this, we find, as before,

$$\tau = \frac{3}{2\vartheta^2} \cdot \frac{a^2}{\nu}.\Big]^*$$

December 11*th*, 1884.

J. W. L. GLAISHER, Esq., F.R.S., President, in the Chair.

The Auditor (Captain MacMahon) made his report; a vote of thanks, moved by Mr. S. Roberts, and seconded by Sir J. Cockle, was unanimously accorded to him for his services.

The Treasurer's Report, read at the November meeting, was then adopted.

The Rev. T. C. Simmons, M.A., Christ's College, Brecon, and Mr. W. J. Ibbetson, M.A., Clare College, Cambridge, were elected Members.

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* [This note received March 16th, 1885. Prof. Lamb sends the following corrections of former papers:

Proceedings Lond. Math. Soc.—T. 14, p. 304, line 13. The sign of the last term should be $-$.

T. 15, p. 148, line 23, in the value of M, *read* I *for* I^2.

,, p. 149, in the last foot-note, *read* "in the particular case where the inductive susceptibility of B is so small, &c."]

Mr. Tucker read a paper On a Group of Circles connected with the Nine-Points Circle, considered as the Locus of the Intersections of Orthogonal Simson Lines, and parts of a paper by Mr. R. A. Roberts, entitled Notes on the Plane Unicursal Quartic.

Communications were also made by the Treasurer, Mr. G. Heppel, and the President.

The following presents were received :—

"Royal Society—Proceedings," Vol. xxxvii., No. 233.

"Nautical Almanac," for 1888.

"Educational Times," for December.

"A Synopsis of Elementary Results in Pure and Applied Mathematics, containing Propositions, Formulæ, and Methods of Analysis, with abridged Demonstrations," by G. S. Carr, M.A., Vol. i., Sections x., xi., and xii., 8vo; London, 1884.

"Beiblätter zu den Annalen der Physik und Chemie," Band 8, St. 11; Leipzig, 1884.

"Bulletin de la Société Mathématique de France," T. xii., No. 4; Paris, 1884.

"Bulletin des Sciences Mathématiques et Astronomiques," S. 2, T. viii.; Paris, Dec., 1884.

"Berichte über die Verhandlungen der Königlich-Sächsischen Gesellschaft der Wissenschaften zu Leipzig,—Mathematisch-physische Classe," 1883, 8vo; Leipzig, 1884.

"Transactions of the Connecticut Academy of Arts and Sciences," Vol. vi., Part 1; Newhaven, 1884.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. v., No. 5; Coimbra.

"Acta Mathematica," 5 : 1; Stockholm, 1884.

"Crelle," "Journal für Mathematik," Bd. xcvii., H. 4; Berlin, 1884.

"Ueber die Frage des Weber'schen Gesetzes und Periodicitaetsgesetzes im Gebiete des Zeitsinnes," von G. Th. Fechner, 4to; Leipzig, 1884.

From M. Maurice d'Ocagne :—

"Sur la Droite Moyenne d'un Système de Droites quelconques situées dans un plan" (*Bulletin de la Soc. Math. de France*, Tom. xii., 1884).

"Sur les Transformations centrales des Courbes planes."

"Sur quelques Propriétés générales des Surfaces algébriques de degré quelconque" (*Comptes Rendus*, Nov., 1884).

Notes on the Plane Unicursal Quartic. *By* R. A. Roberts, M.A.

[*Read December* 11*th*, 1884.]

I collect in these notes a few miscellaneous properties of the plane unicursal quartic, most of which, I believe, have not been noticed before. I do not attempt to give any systematic account of this curve, but merely investigate such questions in connection with it as appear most likely to lead to results of some interest.

1. To find the parameters of the nodes of the general unicursal quartic.

The sextic giving these parameters is arrived at in Salmon's *Higher Plane Curves*, Art. 291 (*a*), but the result comes out in a rather complicated form. The following method gives the equation in the form of a single determinant, and has the advantage of being immediately applicable to unicursal curves of any degree.

Suppose the quartic to be determined by the equations

$$\left.\begin{aligned} \rho x &= f_1 = a_1\vartheta^4+b_1\vartheta^3+c_1\vartheta^2+d_1\vartheta+e_1 \\ \rho y &= f_2 = a_2\vartheta^4+b_2\vartheta^3+c_2\vartheta^2+d_2\vartheta+e_2 \\ \rho z &= f_3 = a_3\vartheta^4+b_3\vartheta^3+c_3\vartheta^2+d_3\vartheta+e_3 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(1),$$

the quantities f_1, f_2, f_3 being written without binomial coefficients, then, if L, M are two lines passing through a node, we must have

$$\left.\begin{aligned} L &= u\,(\vartheta-\alpha)\,(\vartheta-\alpha') \\ M &= v\,(\vartheta-\alpha)\,(\vartheta-\alpha') \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where u, v are quadratics in ϑ, and α, α' are the two parameters corresponding to the node. Hence substituting linear expressions in x, y, z for L, M respectively in $Lv-Mu=0$, we get a result which may be written

$$(a\vartheta^2+\beta\vartheta+\gamma)\,x+(a'\vartheta^2+\beta'\vartheta+\gamma')\,y+(a''\vartheta^2+\beta''\vartheta+\gamma'')\,z = 0 \ldots(3),$$

and then

$$\left.\begin{aligned} ax+a'y+a''z &= 0 \\ \beta x+\beta'y+\beta''z &= 0 \\ \gamma x+\gamma'y+\gamma''z &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

represent three lines passing through the node.

We now substitute f_1, f_2, f_3 for x, y, z in (3), and equate to zero the seven coefficients of ϑ^6, ϑ^5, &c., when we get

$$\left.\begin{aligned} aa_1+a'a_2+a''a_3 &= 0 \\ ab_1+a'b_2+a''b_3+\beta a_1+\beta'a_2+\beta''a_3 &= 0 \\ ac_1+a'c_2+a''c_3+\beta b_1+\beta'b_2+\beta''b_3+\gamma a_1+\gamma'a_2+\gamma''a_3 &= 0 \\ ad_1+a'd_2+a''d_3+\beta c_1+\beta'c_2+\beta''c_3+\gamma b_1+\gamma'b_2+\gamma''b_3 &= 0 \\ ae_1+a'e_2+a''e_3+\beta d_1+\beta'd_2+\beta''d_3+\gamma c_1+\gamma'c_2+\gamma''c_3 &= 0 \\ \beta e_1+\beta'e_2+\beta''e_3+\gamma d_1+\gamma'd_2+\gamma''d_3 &= 0 \\ \gamma e_1+\gamma'e_2+\gamma''e_3 &= 0 \end{aligned}\right\} \ldots(5).$$

But the parameters of the node must satisfy the equations obtained by substituting f_1, f_2, f_3 for x, y, z in the equations (4). Hence, takin[g]

the first and last of these equations, and eliminating α, α', &c. by means of (5), we get a determinant which we may write

$$\begin{vmatrix} u_1, & u_2, & u_3, & & & & & & \\ & & & & & & v_1, & v_2, & v_3 \\ a_1, & a_2, & a_3, & & & & & & \\ b_1, & b_2, & b_3, & a_1, & a_2, & a_3, & & & \\ c_1, & c_2, & c_3, & b_1, & b_2, & b_3, & a_1, & a_2, & a_3 \\ d_1, & d_2, & d_3, & c_1, & c_2, & c_3, & b_1, & b_2, & b_3 \\ e_1, & e_2, & e_3, & d_1, & d_2, & d_3, & c_1, & c_2, & c_3 \\ & & & e_1, & e_2, & e_3, & d_1, & d_2, & d_3 \\ & & & & & & e_1, & e_2, & e_3 \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots(6),$$

where
$$u_1 = b_1\vartheta^3 + c_1\vartheta^2 + d_1\vartheta + e_1,$$
$$v_1 = a_1\vartheta^3 + b_1\vartheta^2 + c_1\vartheta + d_1,$$
$$u_2 = \text{\&c.},$$

and the factors corresponding to $\vartheta = 0$ and ∞, respectively, have been divided out.

In exactly the same way we obtain a determinant with $3(n-1)$ rows to determine the parameters of the nodes of the general unicursal curve of the n^{th} degree.

2. By eliminating α, α', &c., between (5), and each pair of the equations (4), we obtain, in the form of determinants, three conics passing through the nodes of the quartic, and, by forming the Jacobian of these conics, we have the equation of the sides of the triangle formed by the nodes.

3. We can find the tangential equation of the nodes of the curve as follows. Let λ, μ, ν be the coordinates of a line passing through a node, then, from (4), we can evidently take

$$\left.\begin{array}{lll} \alpha = l\gamma + m\lambda, & \alpha' = l\gamma' + m\mu, & \alpha'' = l\gamma'' + m\nu \\ \beta = l'\gamma + m'\lambda, & \beta' = l'\gamma' + m'\mu, & \beta'' = l'\gamma'' + m'\nu \end{array}\right\} \quad \ldots\ldots\ldots(7),$$

substituting which values of α, α', &c. in (5), we get seven equations from which we can eliminate γ, γ', γ'', l, m, l', m', when the required result comes out in the form of a determinant $\Delta^2 = 0$, where $\Delta = 0$ gives the three nodes each once.

4. By forming the discriminant of $\lambda f_1 + \mu f_2 + \nu f_3$, we see that the

tangential equation of the curve is of the form $S^3 = T^2$. We can also easily show that this is the case if the equation of the curve referred to the nodal triangle be given.

Writing $U \equiv x^2y^2 + y^2z^2 + z^2x^2 + 2xyz\,(ax + by + cz) = 0$ ....(8),

it is evident, by inversion, that the line $\lambda x + \mu y + \nu z = 0$ will touch the curve if the two conics

$$V \equiv x^2 + y^2 + z^2 + 2ayz + 2bzx + 2cxy,$$

$$V' \equiv \lambda yz + \mu zx + \nu xy,$$

touch one another. Forming then the discriminant of $V + 2kV'$, we get

$$\begin{aligned} 2\lambda\mu\nu k^3 - (\lambda^2 + \mu^2 + \nu^2 - 2a\mu\nu - 2b\nu\lambda - 2c\lambda\mu)\,k^2 \\ + 2\,\{(bc-a)\,\lambda + (ca-b)\,\mu + (ab-c)\,\nu\}\,k \\ + 1 + 2abc - a^2 - b^2 - c^2 = 0 \end{aligned} \quad \text{.........................(9)},$$

and the discriminant of this equation with regard to k will give the tangential equation of U.

We thus find

$$(8\rho^3 + 27\Delta^2\lambda\mu\nu + 9\Delta\rho\Sigma)^2 - (4\rho^2 + 3\Delta\Sigma)^3 = 0 \quad \text{...........(10)},$$

where we have put

$$\lambda^2 + \mu^2 + \nu^2 - 2a\mu\nu - 2b\nu\lambda - 2c\lambda\mu = \Sigma,$$

$$(a-bc)\,\lambda + (b-ca)\,\mu + (c-ab)\,\nu = \rho,$$

$$1 + 2abc - a^2 - b^2 - c^2 = \Delta,$$

and λ, μ, ν are tangential coordinates, and a, b, c parameters.

5. Now, from (10), it is evident that $4\rho^2 + 3\Delta\Sigma = 0$ represents a conic touching the six inflexional tangents, and it is easy to see that Σ is the conic which touches the six tangents at the nodes. Thus we see that these two conics have double contact with each other, the point represented by ρ being the pole of the chord of contact.

6. Since the tangential equation of the curve can be written in the form $S^3 - T^2 = 0$, it follows, from a result given in a paper "On Tangents to a Cubic forming a Pencil in Involution," published in the *Proceedings*, Vol. XIII., p. 25, that there will be a locus of the ninth order from any point of which the tangents to the curve will form a pencil in involution. The complete locus is of the forty-fifth degree, but it must be divisible by this special curve of the ninth degree.

7. To find the locus of the points from which the tangents drawn to the curve have their points of contact on a conic.

We know that the points of contact of the tangents from a point P lie on a cubic passing through the nodes, viz., the polar cubic of P. Now, if six points of intersection of a quartic and a cubic lie on a conic, the remaining six points of intersection must lie on another conic. Hence, since the nodes count doubly as intersections of the two curves, they must be the points of contact of a conic having triple contact with the cubic. But, by a property of the cubic, when this is the case, the points where the sides meet the curve again must lie on a line.

Now, the polar cubic of x', y', z' with regard to U is

$$x'x(y^2+z^2)+y'y(z^2+x^2)+z'z(x^2+y^2)+(ax'+by'+cz')\,xyz$$
$$+(ax+by+cz)(x'yz+y'zx+z'xy)=0 \quad \text{.........}(11).$$

Hence, for the points where x, y, z meet the curve again, we have

$$x=0,\;\; y(z'+bx')+z(y'+cx')=0; \quad y=0,\;\; x(z'+ay')+z(x'+cy')=0;$$
$$z=0,\;\; x(y'+az')+y(x'+bz')=0.$$

Hence, expressing that these points lie on a line, we get

$$(a-bc)\,x(y^2-z^2)+(b-ca)\,y(z^2-x^2)+(c-ab)\,z(x^2-y^2)=0\;\text{...}(12).$$

This, then, is the equation of a cubic such that the six tangents from any point thereof to the quartic have their points of contact on a conic.

8. I proceed to consider what this locus becomes for a few special forms of the equation (8). If in the foregoing equation (8) we have $c=1$, the node xy becomes a cusp, and the cubic (12) becomes divisible by the cuspidal tangent, the remaining factor being the conic $(a-b)(xy-z^2)+(1-ab)\,z(x-y)=0$, which is the locus of points from which the tangents have their five points of contact on a conic passing through the cusp.

Let $a=b=c=0$, then the cubic (12) vanishes identically, and we see that the points of contact of the tangents drawn from any point

to the curve $\qquad x^2y^2+y^2z^2+z^2x^2$ ....(13)

lie on a conic. We can find the equation of this conic as follows. The points of contact of the tangents from x', y', z' evidently lie on

the curve $$\frac{x'}{x^3}+\frac{y'}{y^3}+\frac{z'}{z^3}=0,$$

and, combining this with (13), we get

$$\left.\begin{array}{c}\dfrac{\vartheta}{x'^2}=\left(\dfrac{x}{x'}\right)^3\left(\dfrac{y}{y'}-\dfrac{z}{z'}\right),\quad \dfrac{\vartheta}{y'^2}=\left(\dfrac{y}{y'}\right)^3\left(\dfrac{z}{z'}-\dfrac{x}{x'}\right)\\ \dfrac{\vartheta}{z'^2}=\left(\dfrac{z}{z'}\right)^3\left(\dfrac{x}{x'}-\dfrac{y}{y'}\right)\end{array}\right\}\ldots\ldots(14),$$

where ϑ is indeterminate. We have then

$$\left.\begin{array}{r}\vartheta\left(\dfrac{1}{x'^2}+\dfrac{1}{y'^2}+\dfrac{1}{z'^2}\right)=\left(\dfrac{x}{x'}+\dfrac{y}{y'}+\dfrac{z}{z'}\right)\Delta\\ \vartheta\left(\dfrac{x}{x'^3}+\dfrac{y}{y'^3}+\dfrac{z}{z'^3}\right)=\left(\dfrac{x^2}{x'^2}+\dfrac{y^2}{y'^2}+\dfrac{z^2}{z'^2}+\dfrac{yz}{y'z'}+\dfrac{zx}{z'x'}+\dfrac{xy}{x'y'}\right)\Delta\end{array}\right\}\ldots(15),$$

where $\Delta\equiv\left(\dfrac{x}{x'}-\dfrac{y}{y'}\right)\left(\dfrac{y}{y'}-\dfrac{z}{z'}\right)\left(\dfrac{z}{z'}-\dfrac{x}{x'}\right).$

Hence, eliminating ϑ between the two equations (15), we get the conic

$$(y'^2+z'^2)\,x^2+(z'^2+x'^2)\,y^2+(x'^2+y'^2)\,z^2+y'z'yz+z'x'zx+x'y'xy=0.$$

9. I give here the locus of points whence the tangents to the curve have their points of contact on a conic for three special forms not included under equation (8). For the nodo-tacnodal quartic

$$x^4+y^2z^2+x^2y^2+2xy\,(ayz+bzx+cx^2)=0$$

(see Salmon's *Higher Plane Curves*, Art. 289), the locus is the conic

$$2b\,(1-a^2)\,x^2+2\,(1-a^2)\,zx+(c-ab)\,yz+(a+bc-2ab^2)\,xy=0.$$

For the oscnodal quartic

$$(yz+x^2)^2+2cxy\,(yz+x^2)+y^2\,(x^2+y^2+2hxy+2fyz)=0,$$

we find similarly the line

$$(1-c^2-2f)\,x+(h-cf)\,y=0.$$

For the quartic with a triple point, the locus is found to consist of three right lines passing through that point.

10. To find the relations connecting the parameters of four points on the curve which lie on a line.

If the line $lx+my+nz=0$ meet the curve, we have, from (1),

$$lf_1+mf_2+nf_3\propto\phi\,(\vartheta)=0,$$

where $\phi\,(\vartheta)=(\vartheta-\vartheta_1)\,(\vartheta-\vartheta_2)\,(\vartheta-\vartheta_3)\,(\vartheta-\vartheta_4).$

Now, let a, a' be the parameters of a node, then we have $\phi\,(a)$ and

$\phi(\alpha')$ respectively proportional to $lf_1+mf_2+nf_3$ and $lf'_1+mf'_2+nf'_3$. But $\frac{f_1}{f'_1}=\frac{f_2}{f'_2}=\frac{f_3}{f'_3}$ for a node obviously.

Hence we get

$$\phi(\alpha)-k_1\phi(\alpha')=0,\quad \phi(\beta)-k_2\phi(\beta')=0,\quad \phi(\gamma)-k_3\phi(\gamma')=0 \ldots(16),$$

where β, β', γ, γ' are the parameters of the two other nodes, and k_1, k_2, k_3 are constants. These three relations are evidently not independent, but are only equivalent to two. There is, in fact, an identical linear relation of the form

$$l\{\phi(\alpha)-k_1\phi(\alpha')\}+m\{\phi(\beta)-k_2\phi(\beta')\}+n\{\phi(\gamma)-k_3\phi(\gamma')\}=0 \quad \ldots\ldots\ldots\ldots(17).$$

By referring the curve to the triangle formed by the nodes, we find

$$k_1=\frac{(\alpha-\beta)(\alpha-\beta')(\alpha-\gamma)(\alpha-\gamma')}{(\alpha'-\beta)(\alpha'-\beta')(\alpha'-\gamma)(\alpha'-\gamma')},\quad k_2=\&c.$$

11. In the same way, if we seek the intersection of the curve with a conic, we get

$$\phi(\alpha)-k_1^2\phi(\alpha')=0,\quad \phi(\beta)-k_2^2\phi(\beta')=0,\quad \phi(\gamma)-k_3^2\phi(\gamma')=0 \ldots(18),$$

where

$$\phi(t)=(t-\vartheta_1)(t-\vartheta_2)(t-\vartheta_3)(t-\vartheta_4)(t-\vartheta_5)(t-\vartheta_6)(t-\vartheta_7)(t-\vartheta_8).$$

In this case the three relations (18) are evidently independent. Hence, if a conic have quartic contact with the curve, the four points of contact must satisfy the equations

$$\phi(\alpha)\pm k_1\phi(\alpha')=0,\quad \phi(\beta)\pm k_2\phi(\beta')=0,\quad \phi(\gamma)\pm k_3\phi(\gamma')=0 \ldots(19),$$

where now $\quad \phi(t)=(t-\vartheta_1)(t-\vartheta_2)(t-\vartheta_3)(t-\vartheta_4)$.

We cannot take two negative signs in (19), as then, from (16) and (17), the points would necessarily lie on a line; but we may take two positive and the other negative, and thus we get three distinct systems of conics having quartic contact with the curve. By taking three positive signs in (19), we get another system of conics, which are evidently perfectly symmetrical with regard to the three nodes.

12. Now, if z be the line joining two of the nodes, it is evident that the curve can be written in the form $S^2-z^2S'=0$, where S and S' are conics, and then

$$\vartheta^2z^2+2\vartheta S+S'=0$$

will represent a conic having quartic contact with the curve. We

thus have three different systems of such conics corresponding to each pair of nodes, which are evidently the same as those we have come upon above.

13. Again, suppose the quartic to be written in the form

$$U \equiv ay^2z^2+bz^2x^2+cx^2y^2+2xyz\,(fx+gy+hz) = 0 \quad\text{.........}(20),$$

then it is evident that the envelope of the system of conics

$$Al^2x^2+Bm^2y^2+Cn^2z^2+2Fmnyz+2Gnlzx+2Hlmxy = 0 \quad\text{......}(21),$$

subject to the condition $\qquad l+m+n=0 \quad\text{.........................}(22),$

is the curve U, where A, B, &c. are the differentials with regard to a, b, &c., of $\Delta \equiv abc+2fgh-af^2-bg^2-ch^2$. The conic (21) evidently belongs to the symmetrical system obtained by taking all the signs positive in (19).

Since the conic (21) is transformed into a fixed conic by substituting x, y, z for lx, my, nz, it follows that any covariant of the triangle of reference and the conic (21) will be transformed into fixed curves by the same substitution. Hence, from the condition (22), it appears that the polars of the nodes with regard to (21) pass through fixed points. Also the poles of the sides with regard to (21) lie on conics circumscribing the triangle.

14. If we suppose two of the nodes to become the circular points at infinity I, J, the curve will become a bicircular quartic with a node, and the conic (21) is then the result of transforming a fixed conic by the substitution of u, v for $\frac{lu}{n}$, $\frac{mv}{n}$, respectively, where u, v are the lines joining the node to the points I, J. Now, this transformation is equivalent to turning the curve about the origin and altering the radii vectores in a constant ratio, the dilätation and the angle through which the figure is turned being connected by a certain relation determined by (22). Hence, in this case, it appears that the axes, asymptotes, tangents at the vertices and directrices of (21) will all turn about fixed points, and the centre, foci, vertices, &c. will move on fixed circles passing through the node. It is evident also that the eccentricity of (21) is given.

15. By taking the envelope of the tangential equation of (21), viz.,

$$\frac{a\lambda^2}{l^2}+\frac{b\mu^2}{m^2}+\frac{c\nu^2}{n^2}+\frac{2f\mu\nu}{mn}+\frac{2g\nu\lambda}{nl}+\frac{2h\lambda\mu}{lm} = 0,$$

subject to the condition (22), we get the tangential equation of the quartic, and the invariants of the biquadratic in $\frac{l}{m}$ will give the

envelopes of lines divided harmonically and equi-anharmonically by the curve.

16. If a tangent to a conic U is homographic with a tangent to a conic V, their intersection will lie on a trinodal quartic having quartic contact with U and V. For a tangent to U, and a homographic tangent to V, can be written in the forms

$$\vartheta^2 P + 2\vartheta R + Q = 0, \quad \vartheta^2 P' + 2\vartheta R' + Q' = 0,$$

and the result of eliminating ϑ between these equations is

$$\left.\begin{array}{ll} & 4\,(RQ'-R'Q)\,(PR'-P'R)-(PQ'-P'Q)^2 = 0 \\ \text{or} & (PQ'+P'Q-2RR')^2-4\,(PQ-R^2)\,(P'Q'-R'^2) = 0 \end{array}\right\} \ldots\ldots (23),$$

which evidently represent a quartic having quartic contact with $PQ-R^2$ and $P'Q'-R'^2$, of which the three points determined by the equations

$$\frac{P}{P'} = \frac{Q}{Q'} = \frac{R}{R'} \ldots\ldots (24)$$

are nodes. It is easy to show that the conics U, V belong to the system (21); for the variable conics of the same system as U, V are evidently

$$(P+kP')\,(Q+kQ')-(R+kR')^2 = 0 \ldots\ldots (25).$$

But, if we take the nodes as triangle of reference, we must write, from (24),

$$\left.\begin{array}{ll} P = ax + \beta y + \gamma z, & P' = lax + m\beta y + n\gamma z \\ Q = a'x + \beta' y + \gamma' z, & Q' = la'x + m\beta' y + n\gamma' z \\ R = a''x + \beta'' y + \gamma'' z, & R' = la''x + m\beta'' y + n\gamma'' z \end{array}\right\} \ldots (26);$$

from which it appears that the conic (25) will be transformed into a fixed conic by substituting $\frac{x}{1+kl}$, $\frac{y}{1+km}$, $\frac{z}{1+kn}$ for x, y, z respectively.

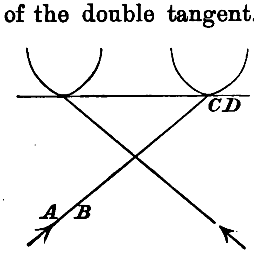
17. In the same way as in § 10, we can show that, if a curve of the n^{th} degree meet the quartic, we must have

$$\phi\,(a)-k_1^n\phi\,(a') = 0, \quad \phi\,(\beta)-k_2^n\phi\,(\beta') = 0, \quad \phi\,(\gamma)-k_3^n\phi\,(\gamma') = 0,$$

where $\qquad \phi\,(\vartheta) = (\vartheta-\vartheta_1)\,(\vartheta-\vartheta_2) \ldots\ldots (\vartheta-\vartheta_{4n}).$

It may be observed that this method is also applicable to unicursal curves of any degree, and will give all the relations connecting the parameters of the points of intersection with a curve whose degree is assigned.

18. If we attempted to find whether it were possible to inscribe an infinite number of closed polygons in a unicursal quartic which should also be circumscribed about the curve, we should be led to expect the well-known relation of the second degree connecting the sum and products of the parameters of the points where a tangent meets the curve again. The curve, then, would have to be of the fourth class. Such curves are: (1) the nodo-bicuspidal quartic, (2) the quartic with a triple point at which the tangents coincide, (3) the quartic with a cusp and a ramphoid cusp; but the two latter curves have no indeterminate constant, which must exist and be determined afterwards by a condition depending on the number of sides of the polygon. Let us consider, then, the nodo-bicuspidal quartic which may be written
$$(xy+yz+zx)^2-m^2z^2xy = 0 \text{......................} (27).$$
This equation is satisfied by assuming
$$x = 1+\vartheta^2-m\vartheta, \quad y = \vartheta^2(1+\vartheta^2-m\vartheta), \quad z = -\vartheta^2 \text{......} (28),$$
and then it is easy to see that the parameters of the points where any line meets the curve are connected by the relations
$$\Sigma_4^1 \vartheta = m, \quad \Sigma_4^1 \frac{1}{\vartheta} = m \text{......................} (29).$$
Hence, putting $\vartheta_4 = \vartheta_3$, and eliminating ϑ_3, we get
$$(\vartheta_1+\vartheta_2)^2-m(\vartheta_1+\vartheta_2)(1+\vartheta_1\vartheta_2)+(m^2-4)\vartheta_1\vartheta_2 = 0 \text{......} (30),$$
which is of the form mentioned above. For the triangle the value of m which we find is irrelevant, and therefore no such triangles can be described for any curve of the form (27). For the quadrilateral, we get $m^2 = -2$, and for this curve it can be shown then that the lines joining the points of contact of opposite sides intersect at the node.* The case $m^2 = -2$ is a very interesting result; it means that the tangent at each cusp passes through a point of contact of the double tangent.



The line in question taken twice is a degenerate form of quadrilateral, viz.,

AB quâ line joining two coincident points at cusp,

BC quâ tangent at cusp,

CD quâ double tangent,

and *DA quâ* tangent at cusp—are each of them a tangent to the curve.

* I am indebted to Professor Cayley for the following remarks on this case.

For x, y, z, writing $1+x$, $1-x$, $\frac{1}{2}y$, and for m^2 writing $-4k^2$, we have the Cartesian equation,

$$(x^2+y-1)^2-k^2y^2(x^2-1)=0,$$

where the curve is as in figure, and of course $k^2=\frac{1}{2}$ is the value for the case of the quadrilateral.

To find the values of m corresponding to polygons of a greater number of sides, we equate to zero certain determinants formed out of the coefficients of the expansion of the function

$$\{\lambda^3+(3-2k)\lambda^2+(k^2-3k+3)\lambda+1-k\}^{\frac{1}{2}},$$

in powers of λ, where $k=\frac{1}{4}m^2$, in the same way as has been done by Professor Cayley for polygons inscribed in one conic and circumscribed about another. (*Philosophical Magazine*, Vol. VI., p. 99.)

19. I now proceed to discuss some properties of the unicursal quartic with a triple point. If we suppose the quartic to be given by the equations (1), I find the condition that the curve should have a triple point as follows. If the curve have a triple point, it is evident that we must have

$$P+\vartheta Q=0 \quad\text{..............................(31)},$$

where P and Q are some pair of lines passing through the triple point. Hence, putting

$$P=\alpha x+\beta y+\gamma z,\quad Q=\alpha' x+\beta' y+\gamma' z,$$

we must have, identically, from (31),

$$(\alpha+\vartheta\alpha')f_1+(\beta+\vartheta\beta')f_2+(\gamma+\vartheta\gamma')f_3=0 \quad\text{............(32)}.$$

Equating then the coefficients of ϑ^5, ϑ^4, &c., in (32), to zero, and eliminating α, β, γ, α', β', γ' linearly, we get

$$\begin{vmatrix} a_1, & a_2, & a_3, & & & \\ b_1, & b_2, & b_3, & a_1, & a_2, & a_3 \\ c_1, & c_2, & c_3, & b_1, & b_2, & b_3 \\ d_1, & d_2, & d_3, & c_1, & c_2, & c_3 \\ e_1, & e_2, & e_3, & d_1, & d_2, & d_3 \\ & & & e_1, & e_2, & e_3 \end{vmatrix} = 0 \quad\text{..................(33)}.$$

Also, solving for α, β, γ from five of these equations, and substituting in $\alpha f_1+\beta f_2+\gamma f_3=0$, we get the cubic which determines the three parameters of the triple point, as the coefficient of ϑ^4 vanishes.

20. To find the relations connecting the parameters ϑ_1, ϑ_2, ϑ_3, ϑ_4 of four collinear points on the curve.

If we take the lines x, y passing through the triple point, we may represent the curve thus

$$\rho x = \Theta_3, \quad \rho y = \vartheta\Theta_3, \quad \rho z = \Theta_4 \quad \ldots\ldots\ldots\ldots\ldots\ldots(34),$$

where Θ_3, Θ_4 are polynomials in ϑ of the third and fourth degrees, respectively. For the points, then, where the line $\lambda x + \mu y + \nu z = 0$ meets the curve, we have

$$(\lambda + \mu\vartheta)\,\Theta_3 + \nu\Theta_4 = 0.$$

Hence, if α, β, γ are the roots of Θ_3, we readily find

$$\frac{\phi(\alpha)}{l} = \frac{\phi(\beta)}{m} = \frac{\phi(\gamma)}{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(35),$$

where $\phi(t) = (t-\vartheta_1)(t-\vartheta_2)(t-\vartheta_3)(t-\vartheta_4)$,

and l, m, n are the values of Θ_4 corresponding to α, β, γ respectively.

21. From (35) we can easily find the relation connecting the parameters of the points where a tangent meets the curve again. Putting $\vartheta_4 = \vartheta_3$, and eliminating ϑ_3, we get

$$\frac{(\beta-\gamma)\sqrt{l}}{\sqrt{\{(\alpha-\vartheta_1)(\alpha-\vartheta_2)\}}} + \frac{(\gamma-\alpha)\sqrt{m}}{\sqrt{\{(\beta-\vartheta_1)(\beta-\vartheta_2)\}}} + \frac{(\alpha-\beta)\sqrt{n}}{\sqrt{\{(\gamma-\vartheta_1)(\gamma-\vartheta_2)\}}} = 0 \quad \ldots\ldots\ldots\ldots(36).$$

From this relation it is easy to show that it is impossible to inscribe a triangle in the quartic which shall be also circumscribed about the curve. For, if we put

$$P_\alpha = \frac{(\beta-\gamma)\sqrt{l}}{\sqrt{\{(\alpha-\vartheta_1)(\alpha-\vartheta_2)(\alpha-\vartheta_3)\}}}, \quad P_\beta = \&c.,$$

we get, for a triangle, from (36),

$$\left.\begin{aligned} P_\alpha\sqrt{(\alpha-\vartheta_1)} + P_\beta\sqrt{(\beta-\vartheta_1)} + P_\gamma\sqrt{(\gamma-\vartheta_1)} &= 0 \\ P_\alpha\sqrt{(\alpha-\vartheta_2)} + P_\beta\sqrt{(\beta-\vartheta_2)} + P_\gamma\sqrt{(\gamma-\vartheta_2)} &= 0 \\ P_\alpha\sqrt{(\alpha-\vartheta_3)} + P_\beta\sqrt{(\beta-\vartheta_3)} + P_\gamma\sqrt{(\gamma-\vartheta_3)} &= 0 \end{aligned}\right\} \quad \ldots\ldots(37),$$

which equations are manifestly inconsistent, as one of them, being cleared of radicals, gives a quadratic for ϑ.

22. To show that, if a triangle be inscribed in the curve, the lines

joining the triple point to the points where the sides meet the curve again form a pencil in involution.

Let a, b, c be the parameters of the vertices, and u_1, u_2, u_3 the three quadratics which determine the parameters of the points where the sides meet the curve again, then x, y, z being the sides of the triangle, we may write

$$x = (\vartheta-b)(\vartheta-c)\,u_1,\ y = (\vartheta-c)(\vartheta-a)\,u_2,\ z = (\vartheta-a)(\vartheta-b)\,u_3 \ldots (38).$$

Now we have seen that, when the curve has a triple point, we must have

$$(a\vartheta+a')(\vartheta-b)(\vartheta-c)\,u_1+(\beta\vartheta+\beta')(\vartheta-c)(\vartheta-a)\,u_2 \\ +(\gamma\vartheta+\gamma')(\vartheta-a)(\vartheta-b)\,u_3 = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots (39),$$

identically. But, putting $\vartheta = a, b, c$ successively in (39), we get

$$\frac{a'}{a} = -a,\quad \frac{\beta'}{\beta} = -b,\quad \frac{\gamma'}{\gamma} = -c,$$

and then each term becomes divisible by $(\vartheta-a)(\vartheta-b)(\vartheta-c)$. We see thus that u_1, u_2, u_3 must be connected by a linear relation, and, therefore, the corresponding parameters form a system in involution.

We can show, conversely, that if the parameters of the points where the sides of a triangle inscribed in a unicursal quartic meet the curve again form a system in involution, then the curve must have a triple point. For, if we have $lu_1+mu_2+nu_3 = 0$, identically in (38), we get

$$lx\,(\vartheta-a)+my\,(\vartheta-b)+nz\,(\vartheta-c) = 0,$$

which shows that the curve has a triple point whose coordinates are

$$\frac{b-c}{l},\quad \frac{c-a}{m},\quad \frac{a-b}{n}.$$

Hence, if we suppose two sides of an inscribed triangle to touch the curve, the lines joining the triple point to the points of contact and the points where the third side meets the curve again are harmonically connected. Hence also, as we have seen before, it is impossible to describe a triangle which shall be simultaneously inscribed in and circumscribed about the curve.

Again, we can deduce the following theorem: If a quartic with a triple point be described through the vertices of a fixed triangle, and through fixed pairs of points on the sides, the locus of the triple point is a cubic curve, of which the fixed points on the sides are pairs of corresponding points.

23. If the pairs of points on the sides are at the extremities of the

diagonals of a complete quadrilateral, we know that the lines joining these points to an arbitrary point form a pencil in involution. Hence it would appear that a quartic can be described through the six points of intersection of the sides of a quadrilateral, and the vertices of the triangle formed by the diagonals, so as to have an arbitrary point for a triple point. We can verify this result independently as follows: Let x, y, z be the diagonals, and $x \pm y \pm z = 0$ the sides of the quadrilateral, then the equation of the quartic must be of the form

$$\phi \equiv ayz\,(y^2-z^2)+bzx\,(z^2-x^2)+cxy\,(x^2-y^2)+3xyz\,(lx+my+nz)=0 \quad \text{...........}(40).$$

Now, if a curve have a triple point, the six second differentials must vanish for this point; hence, if x, y, z is the triple point, we have, from (40),

$$cxy-bzx+lyz=0,$$

$$ayz-cxy+mzx=0, \quad bzx-ayz+nxy=0,$$

$$\left.\begin{aligned} a\,(y^2-z^2)+lx^2+2mxy+2nzx&=0\\ b\,(z^2-x^2)+my^2+2lxy+2nyz&=0\\ c\,(x^2-y^2)+nz^2+2lzx+2myz&=0 \end{aligned}\right\} \quad \text{..............}(41).$$

All these equations are satisfied by the values

$$\left.\begin{aligned} a=x^3,\quad b=y^3,\quad c=z^3\\ l=x\,(y^2-z^2),\quad m=y\,(z^2-x^2),\quad n=z\,(x^2-y^2) \end{aligned}\right\} \quad \text{.........}(42),$$

showing that the triple point can be assumed arbitrarily.

24. For a quartic with a triple point, the invariants A and B (Salmon's *Higher Plane Curves*, Arts. 293, 294) vanish, and we can verify that this is the case, when a, b, &c. have the values (42). Calculating A and B for the quartic (40), we get

$$\left.\begin{aligned} A&=-12\,(lmn+bcl+cam+abn)\\ B&=(lmn+bcl+cam+abn)^2 \end{aligned}\right\} \quad \text{..............}(43),$$

but these vanish when we put in for a, b, &c., from (42). It may be observed that it is not possible to write a general quartic in the form (40); for, from (43), we see that the curve must satisfy the invariant relation $A^2=144B$.

25. To show that, if a triangle be inscribed in a quartic with a triple point so as to have its sides divided harmonically by the curve, the points where the sides meet the curve again must be at the extremities of the diagonals of a quadrilateral.

If the sides x, y, z of an inscribed triangle are divided harmonically by a quartic U, we have

$$U \equiv yz\,(b_1 y^2 + c_1 z^2) + zx\,(a_2 x^2 + c_2 z^2) + xy\,(a_3 x^2 + b_3 y^2) + 3xyz\,(lx + my + nz) = 0.$$

Calculating then the invariants A and B, we find

$$A = 12p + 4q, \quad B = (p-q)^2 \qquad (44),$$

where $\quad p = lb_3c_2 + mc_1a_3 + nb_1a_2 - lmn, \quad q = a_2b_3c_1 + a_3b_1c_2.$

Now, we have seen that, when the curve has a triple point, A and B vanish; hence, from (44), we get $p = q = 0$, but $p = 0$ is the condition that the points on the sides should be at the extremities of the diagonals of a quadrilateral. I now proceed to show that, if such a triangle exist, the quartic must satisfy a special condition, and that there are then an infinite number of these triangles inscribed in the curve. If we take ∞, 0 as the parameters of the double points of the involution determined by the points where the sides meet the curve again, we may write the equations (38) as follows:

$$\left.\begin{aligned} \rho lx = (\vartheta^2 - k_1^2)(\vartheta - b)(\vartheta - c), \quad \rho my = (\vartheta^2 - k_2^2)(\vartheta - c)(\vartheta - a) \\ \rho nz = (\vartheta^2 - k_3^2)(\vartheta - a)(\vartheta - b) \end{aligned}\right\} \ldots (45).$$

But, if x divide the curve harmonically, we must have $k_1^2 = bc$, and similarly for y and z, $k_2^2 = ca$, $k_3^2 = ab$. Now, if we determine l, m, n, so that, when we put $x = 0$, $y = 0$, $z = 0$, we get $y^2 - z^2 = 0$, $z^2 - x^2 = 0$, $x^2 - y^2 = 0$, respectively, we find

$$l = \frac{(a-b)(a-c)}{a^{\frac{3}{2}}}, \quad m = \frac{(b-a)(b-c)}{b^{\frac{3}{2}}}, \quad n = \frac{(c-a)(c-b)}{c^{\frac{3}{2}}}.$$

Eliminating, then, ϑ between the equations (45), we obtain

$$(\alpha x + \beta y + \gamma z)^3 \left(\frac{x}{\alpha^3} + \frac{y}{\beta^3} + \frac{z}{\gamma^3}\right) = \left(\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma}\right)^3 (\alpha^3 x + \beta^3 y + \gamma^3 z) \ldots (46),$$

where we have put $a = \alpha^2$, $b = \beta^2$, $c = \gamma^2$. Now this equation (46) evidently does not represent the general quartic with a triple point, but one in which two inflexional tangents meet on the curve. If we seek the parameters of the triple point in this case, we find

$$\vartheta^3 - abc = 0 \qquad (47),$$

which shows that the double lines of the involution are absolutely

fixed, and coincide with the Hessian lines of the tangents at the triple point.

It may be observed that, when the curve is written in the form (40), the tangents at the vertices are $cy-bz=0$, $az-cx=0$, $bx-ay=0$. Thus we see that these lines always pass through a point.

26. I give an independent proof that the problem, to inscribe a triangle in the quartic with a triple point, so that the sides may be divided harmonically by the curve, is either indeterminate or impossible.

Let the curve referred to the triangle formed by the triple point and the points of contact of a double tangent be written

$$x^2y^2-z\,(y-ax)(y-bx)(y-cx)=0,$$

then we may take $x=f$, $y=\vartheta f$, $z=\vartheta^2$, where $f=(\vartheta-a)(\vartheta-b)(\vartheta-c)$, and then, from (35), if any right line meet the curve, we have

$$\frac{\phi(a)}{a^2}=\frac{\phi(b)}{b^2}=\frac{\phi(c)}{c^2} \quad \text{......................}(48).$$

Now, if ϑ_1, ϑ_2, ϑ_3, ϑ_4 are harmonic, we have

$$2\,(\vartheta_1\vartheta_2+\vartheta_3\vartheta_4)=(\vartheta_1+\vartheta_2)(\vartheta_3+\vartheta_4) \quad \text{..............}(49).$$

Eliminating then ϑ_3 and ϑ_4 between (48) and (49), we get

$$\begin{aligned}\frac{a^2}{(a-b)(a-c)}\,&\frac{\{2\,(\vartheta_1\vartheta_2+bc)-(b+c)(\vartheta_1+\vartheta_2)\}}{(a-\vartheta_1)(a-\vartheta_2)}\\ +\frac{b^2}{(b-a)(b-c)}\,&\frac{\{2\,(\vartheta_1\vartheta_2+ca)-(c+a)(\vartheta_1+\vartheta_2)\}}{(b-\vartheta_1)(b-\vartheta_2)}\\ +\frac{c^2}{(c-a)(c-b)}\,&\frac{\{2\,(\vartheta_1\vartheta_2+ab)-(a+b)(\vartheta_1+\vartheta_2)\}}{(c-\vartheta_1)(c-\vartheta_2)}=0 \quad \text{...}(50).\end{aligned}$$

Let us suppose now a conic defined by the equations

$$x=(b-c)^2(\vartheta-a)^2,\quad y=(c-a)^2(\vartheta-b)^2,\quad z=(a-b)^2(\vartheta-c)^2,$$

i.e.,

$$\sqrt{x}+\sqrt{y}+\sqrt{z}=0 \quad \text{......................}(51),$$

then, if x, y, z are the coordinates of the intersection of the tangents at the points ϑ_1, ϑ_2, the equation (50) will become

$$\begin{aligned}a^2(b-c)^4\,yz\,(y+z-x)+b^2(c-a)^4\,zx\,(z+x-y)&\\ +c^2(a-b)^4\,xy\,(x+y-z)=0& \quad \text{........}(52).\end{aligned}$$

Hence the problem becomes, to circumscribe triangles about the conic (51) which shall be inscribed in the cubic (52). Now I have sho[redacted]

in § 12 of a paper published in the *Proceedings*, Vol. xv., p. 4, that when it is possible to circumscribe an infinite number of triangles about a conic so as to be inscribed in a cubic, then the points where the sides meet the cubic again must lie on a line, and this line must touch the conic. But this line for the cubic (52) is $x+y+z=0$, which cannot possibly touch the conic (51). If, however, we have

$$a^2(b-c)^4+b^2(c-a)^4+c^2(a-b)^4=0 \qquad (53),$$

the cubic (52) becomes divisible by $x+y+z$, the remaining factor being the conic

$$a^2(b-c)^4yz+b^2(c-a)^4zx+c^2(a-b)^4xy=0;$$

and, since this conic circumscribes an infinite number of triangles circumscribed about (51), the problem then becomes indeterminate. We can verify that (53) is the condition that two inflexional tangents should meet on the curve. Let ϑ_1, ϑ_2 be the parameters of the inflexions, and ϕ the parameter of the intersection of the corresponding tangents; then, from (48), we have

$$\frac{a^2}{(a-\vartheta_1)^3(a-\phi)}=\frac{b^2}{(b-\vartheta_1)^3(b-\phi)}=\frac{c^2}{(c-\vartheta_1)^3(c-\phi)},$$

$$\frac{a^2}{(a-\vartheta_2)^3(a-\phi)}=\frac{b^2}{(b-\vartheta_2)^3(b-\phi)}=\frac{c^2}{(c-\vartheta_2)^3(c-\phi)},$$

whence, eliminating ϑ_1, ϑ_2, ϕ, we get (53).

27. From the theorem in § 21, it is easy to deduce that it is possible to inscribe an infinite number of triangles in the curve so that the pairs of points where the sides meet the curve again may be conjugate with regard to a fixed pair of lines passing through the triple point. When the fixed lines are the Hessian lines of the triple tangents, we see, from (47), that the triangles are those considered in the preceding paragraph.

[It has been pointed out to me by Professor Cayley that Clebsch (*Crelle*, t. 64, p. 64) has arrived at seven distinct systems of conics which have quartic contact with the plane unicursal quartic, whereas I have arrived at but four systems. Three of these systems appear to have been obtained by considering the cases of two negative and one positive sign in equations (19) of the text. But, from the identical relation (17), we see that these cases cannot exist. A few other statements of Clebsch, in the paper referred to above, seem to require similar modifications.]

January 8th, 1885.

J. W. L. GLAISHER, Esq., F.R.S., President, in the Chair.

Mr. F. R. Barrell, B.A., late Scholar of Pembroke College, Cambridge; Mr. S. O. Roberts, B.A., Scholar of St. John's College, Cambridge; and Mr. M. N. Dutt, B.A., Calcutta University, Professor of Mathematics, St. Stephen's College, Delhi,—were elected members. The Rev. T. C. Simmons was admitted into the Society.

The following communications were made:—

The Differential Equations of Cylindrical and Annular Vortices, Prof. M. J. M. Hill.

On Criticoids, Rev. R. Harley.

Multiplication of Symmetric Functions, Captain MacMahon.

Note on Symmetrical Determinants, Mr. Buchheim.

The President (Mr. Walker taking the chair) communicated Results in Elliptic Functions.

Mr. Tucker communicated a paper (supplementary) on Limits of Multiple Integrals, by Mr. MacColl, and read a Note by Prof. Cayley on the Binomial Equation $x^p-1=0$, Quinquisection (Second Note).

The following presents were received:—

"Annali di Matematica," Serie II., Tomo XII., Fasc. 4°; Dec., 1884.

"Educational Times," for January, 1885.

"Proceedings of the Academy of Natural Sciences of Philadelphia," Part II., May to October, 1884, 8vo; Philadelphia, 1884.

"Nieuw Archief voor Wiskunde," Deel XI.; Amsterdam, 1884.

"Beiblätter zu den Annalen der Physik und Chemie," B. VIII., St. 12.

"Atti della R. Accademia dei Lincei—Transunti," Vol. VIII., F. 16 ed ultimo; Rome, 1884.

The Binomial Equation $x^p-1=0$; *Quinquisection, Second Note.* *By* Prof. CAYLEY.

[*Read January 8th*, 1885.]

In the paper, "The Binomial Equation $x^p-1=0$; Quinquisection," *Proc. Lond. Math. Soc.*, t. 12 (1880), pp. 15–16, I considered for an exponent $p=5n+1$, the five periods X, Y, Z, W, T connected by the equations

$$\begin{array}{r|ccccc} & X, & Y, & Z, & W, & T \\ \hline X^2 = & a, & b, & c, & d, & e \\ XY = & f, & g, & h, & i, & j \\ XZ = & k, & l, & m, & n, & o \end{array}$$

and the equations deduced from these by cyclical permutations of the periods and of the coefficients of each set, but I did not obtain completely even the linear relations connecting the coefficients. I since found, by induction from the examples given in the Table 1, that the coefficients could be expressed linearly in terms of the linearly independent integer numbers α, β, f, k as follows: viz., introducing for convenience the new number θ, such that

$$\alpha+\beta+\theta = \tfrac{1}{5}(p-1),$$

then the expressions in question are

$$\begin{array}{llllll} a,\ b,\ c,\ d,\ e = & -1-2\theta+\alpha+\beta, & -\theta-\alpha-\beta+f, & -\theta-\alpha-\beta+k, & -\alpha-2\beta-k, & -2\alpha-\beta-f, \\ f,\ g,\ h,\ i,\ j = & f, & \theta-\alpha-f, & \alpha, & \beta, & \alpha, \\ k,\ l,\ m,\ n,\ o = & k, & \alpha, & \theta-\beta-k, & \beta, & \beta, \end{array}$$

and I found further that, substituting these values of the coefficients in the 20 quadric relations referred to in the former paper, the 20 relations reduced themselves to two equations only, viz., these were

$$\theta(-2\alpha+\beta+k)+3\alpha^2-\beta^2 \quad + \quad \alpha(f-k-1)-\beta f+f^2-2fk = 0,$$
$$-\theta^2+\theta(3\beta+2k+f) + \alpha^2-\alpha\beta-3\beta^2-\alpha k+\beta(1-f-k)-k^2-2fk = 0.$$

The final result thus is that the coefficients are expressed as functions of the five numbers $\alpha, \beta, f, k, \theta$, connected by the linear equation $\alpha+\beta+\theta = \frac{1}{5}(p-1)$, and the two quadric equations. I remark that formulæ equivalent to these were obtained and proved by Mr. F. S. Carey in his Trinity Fellowship Dissertation, 1884; viz., writing $n = \frac{1}{5}(p-1)$, his formulæ are

$$\begin{array}{llllll} a,\ b,\ c,\ d,\ e = & \alpha-n, & \beta-n, & \gamma-n, & \delta-n, & \epsilon-n, \\ f,\ g,\ h,\ i,\ j = & \beta, & \epsilon, & \rho, & \sigma, & \rho, \\ k,\ l,\ m,\ n,\ o = & \gamma, & \rho, & \delta, & \sigma, & \sigma, \end{array}$$

with the three linear relations

$$\begin{aligned} \alpha+\beta+\gamma+\delta+\epsilon &= n-1, \\ \beta+\epsilon+2\rho+\sigma &= n, \\ \gamma+\delta+\rho+2\sigma &= n, \end{aligned}$$

and the two quadric relations

$$\delta^2+\gamma^2+2\sigma\alpha+(\rho-\sigma)(\delta+\gamma)-2\rho(\rho+\sigma) = (\delta-\gamma)(\beta-\epsilon),$$
$$\beta^2+\epsilon^2+2\rho\alpha+(\sigma-\rho)(\beta+\epsilon)-2\sigma(\rho+\sigma) = (\gamma-\delta)(\beta-\epsilon),$$

the coefficients being thus expressed in terms of the seven numbers

α, β, γ, δ, ϵ, ρ, σ connected by five equations. The equivalence of the two sets of formulæ may be shown without difficulty.

To the Table 2 of the Quintic Equations, given in the paper, may be added the following result from Legendre's "Théorie des Nombres," Ed. 3, t. ii., p. 213,

p	η^5	η^4	η^3	η^2	η	1	
641	1	+1	−256	−564	+5238	−5120	= 0,

calculated by him for the isolated case $p = 641$.

On the Theory of Matrices. *By* Mr. A. Buchheim, M.A.

[*Read Nov.* 13*th*, 1884.]

Introduction.

The methods used in the following paper are essentially, though not historically, an extension of Hamilton's theory of the linear function of a vector, and the simplest way to connect Grassmann's methods with the theory created by Cayley and Sylvester will be to connect them both with Hamilton's investigations.

It is, or ought to be, well known that the linear and vector function of a vector is simply the matrix of the third order. This is obvious from the definition: for, if ρ is any vector, $\sigma = \phi\rho$ is a vector whose constituents are linear functions of ρ's constituents; that is, if

$$\rho = xi + yj + zk, \quad \sigma = x'i + y'j + z'k,$$

we must have the three equations

$$x' = ax + a'y + a''z,$$
$$y' = bx + b'y + b''z,$$
$$z' = cx + c'y + c''z,$$

that is,

$$(x'y'z') = \left(\begin{array}{ccc} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{array}\right)\!\!\!\bigg\backslash\!\!\!\!\!\!\big/ \, x, y, z) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{A}).$$

That is to say, it is the same thing whether we say that $\sigma = \phi\rho$, or

that the constituents of σ are obtained from those of ρ by operating on them with a certain matrix; and we see that in this sense we can identify ϕ with the matrix, and we can say that

$$\phi = \begin{pmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{pmatrix} \quad \text{.........................(B).}$$

Now, in (A), let $\rho = i$, that is, let $(xyz) = (100)$; then

$$(x'y'z') = (abc),$$

that is, $$\sigma = ai + bj + ck,$$

or say, $$\phi i = ai + bj + ck = a.$$

In the same way, we get

$$\phi j = a'i + b'j + c'k = a',$$

$$\phi k = a''i + b''j + c''k = a''.$$

And then
$$\begin{aligned} \phi\rho = \phi(xi + yj + zk) = & \quad (ax + a'y + a''z)\, i \\ & + (bx + b'y + b''z)\, j \\ & + (cx + c'y + c''z)\, k \\ = & \quad x(ai + bj + ck) \\ & + y(a'i + b'j + c'k) \\ & + z(a''i + b''j + c''k) \\ = & \; xa + ya' + za''. \end{aligned}$$

And we can say that (the linear function or matrix) ϕ changes i, j, k into three given vectors a, a', a'', and changes any other vector $xi + yj + zk$ into $xa + ya' + za''$.

Now, on looking at what precedes, it will at once be obvious that we have used none of the special properties of i, j, k: so far as our work is concerned, they might have been any three vectors, provided only that every vector could be expressed in terms of them; and if we call three such vectors asyzygetic, and change the notation, we can say that a linear function, or matrix, changes three given asyzygetic vectors α, β, γ into three given vectors α', β', γ', and changes any vector $x\alpha + y\beta + z\gamma$ into $x\alpha' + y\beta' + z\gamma'$. As regards the word "asyzygetic," I remark that any vector can be expressed in terms of $\alpha\beta\gamma$, provided $S\alpha\beta\gamma$ does not vanish; and we know that $S\alpha\beta\gamma = 0$ is the necessary and sufficient condition that we may have a relation $\lambda\alpha + \mu\beta + \nu\gamma = 0$,

where λ, μ, ν are scalars: it is better to use this as a definition of asyzygetic vectors; viz., three vectors are asyzygetic if they are not connected by a linear relation with scalar coefficients.

If we use the notation of the paper, we can write

$$\phi = \frac{\alpha', \beta', \gamma'}{\alpha, \beta, \gamma},$$

$$\phi\,(x\alpha + y\beta + z\gamma) = x\alpha' + y\beta' + z\gamma'.$$

If
$$(\alpha'\beta'\gamma') = \left(\begin{array}{ccc} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{array}\right)(\alpha\beta\gamma),$$

$$\phi = \left(\begin{array}{ccc} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{array}\right).$$

Before passing on to matrices of any order, I shall give a simple application of the method as an example. I choose the proof of the identical equation (Hamilton's Symbolic Cubic).

It is known (*cf.* Hamilton's *Elements*, § 353, *seq.*) that for any matrix ϕ there are in general three scalars λ, μ, ν, and three vectors α, β, γ, respectively, such that

$$\left.\begin{array}{lll} \phi\alpha = \lambda\alpha & \text{or} & (\phi-\lambda)\,\alpha = 0 \\ \phi\beta = \mu\beta & \text{or} & (\phi-\mu)\,\beta = \\ \phi\gamma = \nu\gamma & \text{or} & (\phi-\nu)\,\gamma = 0 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (\text{C})$$

and that the three vectors α, β, γ are asyzygetic. Let $\rho = x\alpha + y\beta + z\gamma$ be any vector; then

$$\begin{aligned} (\phi-\lambda)\,\rho &= x\,(\phi-\lambda)\,\alpha + y\,(\phi-\lambda)\,\beta + z\,(\phi-\lambda)\,\gamma \\ &= y\,(\phi-\lambda)\,\beta + z\,(\phi-\lambda)\,\gamma, \text{ by (C)}, \\ (\phi-\mu)(\phi-\lambda)\,\rho &= y\,(\phi-\mu)(\phi-\lambda)\,\beta + z\,(\phi-\mu)(\phi-\lambda)\,\gamma \\ &= y\,(\phi-\lambda)(\phi-\mu)\,\beta + z\,(\phi-\lambda)(\phi-\mu)\,\gamma \\ &= z\,(\phi-\lambda)(\phi-\mu)\,\gamma, \text{ by (C)}, \\ (\phi-\nu)(\phi-\mu)(\phi-\lambda)\,\rho &= z\,(\phi-\lambda)(\phi-\mu)(\phi-\nu)\,\gamma \\ &= 0, \text{ by (C)}. \end{aligned}$$

That is, $(\phi-\lambda)(\phi-\mu)(\phi-\nu)\,\rho$ always vanishes; that is,

$$(\phi-\lambda)(\phi-\mu)(\phi-\nu) = 0.*$$

* This result might, of course, have been obtained in one step, and the general theorem is so obtained in the paper. I have preferred the longer form of the proof because it seemed to show the principle involved more clearly.

We have now to extend this theory to matrices of higher orders. It is fairly obvious that, in the case of matrices of the third order, the success of the method depends on the fact that for three variables (x, y, z) we are able to substitute a single vector $(x\alpha+y\beta+z\gamma)$; and the only property of the vector that we have used is the following:

If $x\alpha+y\beta+z\gamma = x'\alpha+y'\beta+z'\gamma$ (α, β, γ being asyzygetic), then

$$x = x', \quad y = y', \quad z = z'.$$

Now, to extend this to sets of more than three letters, take n *units* $e_1, e_2, e_3, \ldots e_n$ (we are not at present concerned with their meaning); and in place of the set of n letters $x_1, x_2, \ldots x_n$ consider the *point*

$$x = x_1 e_1 + x_2 e_2 + x_3 e_3 + \ldots + x_n e_n,$$

and stipulate as before that

$$x_1 e_1 + x_2 e_2 + x_3 e_3 + \ldots + x_n e_n = y_1 e_1 + y_2 e_2 + y_3 e_3 + \ldots + y_n e_n,$$

say $$x = y,$$

shall *mean* $$x_1 = y_1,\ x_2 = y_2 \ldots x_n = y_n.$$

Then we have, for instance,

$$\lambda x + \mu y = (\lambda x_1 + \mu y_1)\, e_1 + \ldots + (\lambda x_n + \mu y_n)\, e_n,$$

where λ, μ are scalars.

We now require the theorem,—Every point can be linearly expressed in terms of any n asyzygetic points. Passing over the word *asyzygetic* for the present, it is easy to see the meaning of the theorem, and to convince oneself of its truth. Let x be any point, and let $\alpha, \beta, \gamma \ldots$ be n given points; then we are to have

$$x = \lambda\alpha + \mu\beta + \nu\gamma + \ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (d),$$

$\lambda, \mu, \nu \ldots$ being scalars, that is

$$\begin{aligned} x_1 e_1 + x_2 e_2 + \ldots + x_n e_n = {} & \lambda\, (\alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n) \\ & + \mu\, (\beta_1 e_1 + \beta_2 e_2 + \ldots + \beta_n e_n) \\ & + \nu\, (\gamma_1 e_1 + \gamma_2 e_2 + \ldots + \gamma_n e_n) \\ & + \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ & \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ = {} & e_1\, (\lambda\alpha_1 + \mu\beta_1 + \nu\gamma_1 + \ldots) \\ & + e_2\, (\lambda\alpha_2 + \mu\beta_2 + \nu\gamma_2 + \ldots) \\ & + \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{aligned}$$

That is, we are to have

$$\left.\begin{array}{l} x_1 = \lambda a_1 + \mu\beta_1 + \nu\gamma_1 + \dots \\ x_2 = \lambda a_2 + \mu\beta_2 + \nu\gamma_2 + \dots \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \end{array}\right\} \quad \dots\dots\dots\dots\dots\dots\text{(D)}.$$

Now we know that these equations determine λ, μ, ν..., if, and only if,

$$\Delta = \begin{vmatrix} a_1 & \beta_1 & \gamma_1 \dots\dots \\ a_2 & \beta_2 & \gamma_2 \dots\dots \\ \vdots & \vdots & \vdots \dots\dots \\ a_n & \beta_n & \gamma_n \dots\dots \end{vmatrix}$$

does not vanish. And therefore, if we say that the points a, β, γ ... are asyzygetic if $\Delta \gtrless 0$, the theorem is proved, and we have also a definition of asyzygetic points. But we can get a better definition: for we know that $\Delta = 0$ is the necessary and sufficient condition that we may be able to solve (D) after putting $x_1 = x_2 \dots = x_n = 0$; and therefore, if we go back to the equation (d) from which (D) was derived, and write, as we obviously may,

$$0 = 0e_1 + 0e_2 + \dots 0e_n,$$

we see that n points a, β, γ ... are *not* asyzygetic if it is possible to satisfy a relation of the form

$$0 = \lambda a + \mu\beta + \nu\gamma + \dots,$$

or, say, if they are connected by a linear relation with scalar coefficients; or, in other words, n points are asyzygetic if they are *not* connected by a linear relation with scalar coefficients. This is the sense in which the word is used in the paper.

Now, suppose we have taken n asyzygetic points $e_1, e_2, \dots e_n$, and have expressed everything in terms of them, and consider the transformation

$$(y_1, y_2, \dots y_n) = \left(\begin{array}{cccc} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{array}\right)(x_1, x_2, \dots x_n),$$

write ϕ to denote the matrix $\| a_{ik} \|$, and denote the transformation by $y = \phi x$.

Now, take $x = e_1$, that is, take

$$(x_1, x_2, \dots x\ ) = (1, 0, \dots 0);$$

then we get $$(y_1, y_2 \dots y_n) = (a_{11}, a_{21} \dots a_{n1}),$$

or $$y = \phi e_1 = a_{11}e_1 + a_{21}e_2 + \dots + a_{n1}e_1 = \alpha_1,\text{*}$$

similarly $$\phi e_2 = a_{12}e_1 + a_{22}e_2 + \dots + a_{n2}e_2 = \alpha_2.$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

Moreover,
$$\begin{aligned}
\phi x = {} & e_1(a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n) \\
& + e_2(a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n) \\
& + \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
& + e_n(a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n) \\
= {} & x_1(a_{11}e_1 + a_{21}e_2 + \dots + a_{n1}e_n) \\
& + x_2(a_{12}e_1 + a_{22}e_2 + \dots + a_{n2}e_n) \\
& + \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
& + x_n(a_{1n}e_1 + a_{2n}e_2 + \dots + a_{nn}e_n) \\
= {} & x_1\alpha_1 + x_2\alpha_2 + \dots + x_n\alpha_n.
\end{aligned}$$

And we see that we can say that the matrix ϕ changes the points of reference, $e_1, e_2 \dots e_n$ into n given points $\alpha_1, \alpha_2 \dots \alpha_n$, and then changes any other point $(x_1e_1 + x_2e_2 + \dots + x_ne_n)$ into $x_1\alpha_1 + x_2\alpha_2 + \dots + x_n\alpha_n$. This is the definition of the matrix used in the paper; the relation between α_1, &c. on the one hand, and the matrix on the other, will be made clear by the following set of equations:

$$(y_1 y_2 \dots y_n) = \left(\begin{array}{cccc} a_{11} & a_{12} & \dots\dots & a_{1n} \\ a_{21} & a_{22} & \dots\dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots\dots & a_{nn} \end{array}\right) \between (x_1, x_2 \dots x_n),$$

$$y = \phi x,$$

$$\alpha_i = \phi e_i,$$

$$(\alpha_1, \alpha_2 \dots \alpha_n) = \left(\begin{array}{cccc} a_{11} & a_{21} & \dots\dots & a_{n1} \\ a_{12} & a_{22} & \dots\dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots\dots & a_{nn} \end{array}\right) \between (e_1, e_2 \dots e_n),$$

* In strict analogy with the rest of the notation, a_1 should of course denote $a_{11}e_1 + a_{12}e_2 + \dots + a_{1n}e_n$; but this inconsistency is unavoidable if we are to keep to the ordinary conventions for matrices. I do not think it need cause any confusion; I have tried to guard against it by using α_1 instead of a_1.

$$\phi x = x_1 a_1 + x_2 a_2 + \ldots + x_n a_n,$$

$$\phi = \frac{a_1,\ a_2\ \ldots\ a_n}{e_1,\ e_2\ \ldots\ e_n}.$$

It remains to add a few words on the multiplication of points. The laws of the multiplication of all points depend on the laws assumed for the *units;* the law assumed by Grassmann is that known as *polar* multiplication; viz., we have $ab = -ba$, $a^2 = 0$, for the original *units* of reference, and then this law holds for all points.* From this, and the associative law, it follows that any product of points vanishes if a point is repeated. We can use this theorem to interpret the products of points. In all that follows, I use geometrical language. The point x is supposed to be the point in a space of $(n-1)$ dimensions, having as its *homogeneous* (multiplanar) coordinates $(x_1, x_2 \ldots x_n)$; and then we can use the following definitions: let α, β be two points, then, if λ is a variable scalar, the point $\alpha + \lambda\beta$ moves on the straight line $\alpha\beta$; if λ, μ are two variable scalars, the point $\alpha + \lambda\beta + \mu\gamma$ moves in the plane $\alpha\beta\gamma$; if λ, μ, ν are scalars, the point $\alpha + \lambda\beta + \mu\gamma + \nu\delta$ moves in the linear space (*three-point*) $\alpha\beta\gamma\delta$; and generally, if $\lambda_1, \lambda_2 \ldots \lambda_{r-1}$ are scalars, the point

$$\Lambda = \alpha + \lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \ldots + \lambda_{r-1} \alpha_{r-1}$$

moves in the r-point $(\alpha, \alpha_1 \ldots \alpha_{r-1})$; since Λ can have a ∞^{r-1} series of positions, depending linearly on $(r-1)$ parameters, it is obvious that an r-point is the same as what Clifford calls an $(r-1)$-flat.

I shall follow Grassmann in enclosing all polar products in square brackets. We have to interpret the product $[\alpha\beta]$: we have

$$\begin{aligned}[(\alpha+\lambda\beta)(\alpha+\lambda'\beta)] &= [\alpha\alpha] + \lambda'[\alpha\beta] + \lambda[\beta\alpha] + \lambda\lambda'[\beta\beta] \\ &= (\lambda' - \lambda)[\alpha\beta].\end{aligned}$$

For $$[\alpha\alpha] = [\beta\beta] = 0, \text{ and } [\beta\alpha] = -[\alpha\beta].$$

Therefore the product is unaltered, to a factor *près*, if for α, β we substitute any two points of the straight line $\alpha\beta$; and it will be altered if we substitute any point not on the straight line (this can easily be verified); thus we see that the product appertains to the straight line, and defines it; we may therefore say that $[\alpha\beta]$ *is* the straight line $\alpha\beta$.† Moreover, we see that

$$\begin{aligned}[\alpha\beta(\alpha+\lambda\beta)] &= [\alpha\beta\alpha] + \lambda[\alpha\beta\beta] \\ &= 0.\end{aligned}$$

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* This law and the commutative $(ab = ba)$ law are the only laws for which this is true; this is proved by Grassmann in his *Ausdehnungslehre*.

† *Cf. Proc. Lond. Math. Soc.*, Vol. xiv., p. 84.

Therefore, the product of two points is the line joining them, and the product of three collinear points vanishes.

In precisely the same way, we can show that the product of three points is the plane containing them, and that the product of four complanar points vanishes; and, generally, the product of r points is the r-*point* determined by them, and the product of $(r+1)$ points contained in the same r-point vanishes.

This last theorem can be put in another form. Suppose the $(r+1)$ points $a_1, a_2 \ldots a_{r+1}$ to be in the same r-point; then, since a_{r+1} is in the r-point $(a_1, a_2 \ldots a_r)$, we have, by definition,

$$a_{r+1} = \lambda_1 a_1 + \lambda_2 a_2 \ldots + \lambda_r a_r.$$

That is, the $(r+1)$ points a are connected by a linear relation, that is, they are not asyzygetic, and, writing r for $r+1$, we can say that the product of r asyzygetic points is the r-point determined by them: if the points are not asyzygetic, the product vanishes. Moreover, it can be proved that, if r points are not asyzygetic, their product will not vanish, and we have, therefore, the important theorem: the necessary and sufficient condition for the existence of a linear relation

$$\sum_1^r \lambda_i a = 0,$$

connecting r points, is $\quad [a_1 a_2 \ldots a\ ] = 0.$

Lastly, I have to remark that the product of n points $x_1, x_2 \ldots x_n$ is

$$\text{Det}\ |\ x_{ik}\ |\ [e_1, e_2 \ldots e_n];$$

and that $[e_1, e_2 \ldots e_n]$ can always be supposed equal to unity.

On p. 241 of the *Ausdehnungslehre* of 1862, Grassmann defines a certain operator, which he calls a quotient: this operator transforms n given points of a space of $(n-1)$ dimensions into n other given points, and then transforms any $(n+1)^{\text{th}}$ point into a determinate point. This operator is, in fact, the general matrix of the n^{th} order; the object of the present paper is to treat the theory of matrices from Grassmann's point of view.* It will be seen that some important parts of the theory are considerably simplified by this treatment. It is hardly necessary to point out that there is not a new theorem in the paper, and that its existence can only be justified, if at all, by the

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* *Cf.* Clifford: "A Fragment on Matrices," ***Math. Papers***, 337.

methods employed. The language and notations of the paper have been explained in the introduction.

1. Take n asyzygetic points, $e_1, e_2 \dots e_n$, and n points corresponding to them, $a_1, a_2 \dots a_n$; then a matrix ϕ of the n^{th} order is defined as an operator, such that
$$\phi e_i = a_i,$$
and that
$$\phi \Sigma c_i e_i = \Sigma c_i \phi e_i = \Sigma c_i a_i,$$
the c being scalars; this matrix can be conveniently written as a fraction
$$\phi = \frac{a_1, a_2 \dots a_n}{e_1, e_2 \dots e_n},$$
or, more simply,
$$\phi = \frac{a_i}{e_i}.$$

We may, if we please, make this notation more definite by adopting a notation of Prof. Cayley's,* and writing
$$\phi = \frac{a_i\,\rvert}{\lvert\, e_i}.$$

Another form is also convenient, and is, in fact, the usual form; let $a_i = \Sigma a_{ji} e_i$, then we write
$$\phi = \left(\begin{array}{cccc} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ \dots & \dots & \dots & \end{array}\right),$$
viz., we have
$$(a_1, a_2, a_3 \dots) = \left(\begin{array}{cccc} a_{11} & a_{21} & a_{31} & \dots \\ a_{12} & a_{22} & a_{32} & \dots \\ a_{13} & a_{23} & a_{33} & \dots \\ \dots & \dots & \dots & \end{array}\right)(e_1, e_2, e_3 \dots),$$
and then
$$\phi = \frac{a_i}{e_i}$$
has the form just given.

2. Two matrices, ϕ, ϕ', are said to be equal if $\phi x = \phi' x$, whatever x

* $\frac{a\,\rvert}{\lvert\, b}\, b = a, \quad b\, \frac{\lvert\, a}{b\,\rvert} = a.$

may be; if e'_i is any set of n asyzygetic points, $\phi = \phi'$ if $\phi e'_i = \phi' e'_i$; for we can express x in the form $\Sigma x'_i e'_i$, and then we have

$$\begin{aligned}\phi x &= \Sigma x'_i \phi e'_i \\ &= \Sigma x'_i \phi' e'_i \\ &= \phi' x.\end{aligned}$$

Hence we can prove that, if $e'_i = \Sigma c_{ij} e_j$ is any asyzygetic set,*

$$\phi = \frac{a_i}{e_i} = \frac{\Sigma c_{ij} a_j}{e'_i},$$

for $$\phi e'_i = \Sigma c_{ij} \phi e_j = \Sigma c_{ij} a_j.$$

Lastly, if $\phi' e_i = \lambda \phi e_i$, where λ is a scalar, we obviously have generally $\phi' x = \lambda \phi x$, or $\phi' = \lambda \phi$, and, if $\phi e_i = \lambda e_i$, $\phi = \lambda$.

5. If we have $$\phi = \frac{a_i}{e_i},$$

$$\chi = \frac{b_i}{a_i},$$

we define the product $\chi\phi$ by the equation

$$\chi\phi = \frac{b_i}{e_i},$$

that is, $$\chi\phi = \frac{b_i|}{|a_i} \cdot \frac{a_i|}{|e_i} = \frac{b_i|}{|e_i}.$$

This product need obviously not be commutative. I proceed to show that it is associative. Let

$$\phi = \frac{a_i}{e_i}, \quad \phi' = \frac{a'_i}{a_i}, \quad \phi'' = \frac{a''_i}{a'_i}.$$

Then $$(\phi''\phi')\,\phi = \left(\frac{a''_i}{a'_i} \cdot \frac{a'_i}{a_i}\right) \frac{a_i}{e_i} = \frac{a''_i}{a_i} \cdot \frac{a_i}{e_i} = \frac{a''_i}{e_i},$$

$$\phi''\,(\phi'\phi) = \frac{a''_i}{a'_i} \left(\frac{a'_i}{a_i} \cdot \frac{a_i}{e_i}\right) = \frac{a''_i}{a'_i} \cdot \frac{a'_i}{e_i} = \frac{a''_i}{e_i}.$$

and therefore the product is associative.

The following formula is important, but no use is made of it in this paper.†

Let $$\phi = \frac{\Sigma a_{ji} e_j}{e_i}, \quad \phi' = \frac{\Sigma b_{ji} e_j}{e_i}.$$

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* *Set* means *set of n points*.

† It is, in fact, the ordinary formula for the multiplication of two matrices.

Then
$$\phi'\phi = \frac{\sum_j e_j \left(\sum_i a_{ik} b_{ji}\right)}{e_k}.$$

We have
$$\phi = \frac{\sum_j b_{ji} e_j}{e_i}$$
$$= \frac{\sum_i a_{ik} \sum_j b_{ji} e_j}{\sum_i a_{ik} e_i} \qquad (k = 1 \dots n)$$
$$= \frac{\sum_j e_j \sum_i a_{ik} b_{ji}}{a_k} \qquad (k = 1 \dots n).$$

Therefore
$$\phi'\phi = \frac{\sum_j e_j \left(\sum_i a_{ik} b_{ji}\right)}{e_k} \qquad (k = 1 \dots n).$$

6. We have now to consider the following problem: Given a matrix ϕ, to find a scalar λ, and a point x, such that
$$\phi x = \lambda x,$$
or
$$(\phi - \lambda)\, x = 0.$$
Let
$$x = \sum_i x_i e_i$$
be the required point, then we have
$$0 = (\phi - \lambda)\, x = \Sigma x_i\, (\phi - \lambda)\, e_i \dots\dots\dots\dots (\text{A}).$$
That is, the n points $(\phi - \lambda)\, e_i$ are asyzygetic; and their product therefore vanishes, that is, λ must satisfy the equation
$$\left[\prod_i (\phi - \lambda)\, e_i\right] = 0 \dots\dots\dots\dots (\text{B}).$$

If this equation be multiplied out, we get an expression $f\lambda\, [e_1 \dots e_n]$, and, as the second factor does not vanish, λ must be a root of $f\lambda = 0$; and then the x_i are obtained (by solving a set of linear equations) as the coefficients of the syzygy (A). If there are s unequal roots of the equation $f\lambda = 0$, we obviously get s points x, one such point appertaining to each root: in particular, if the n roots are all unequal, we get n points. It is possible, however, in every case to get n points appertaining in groups to the different roots of $f\lambda = 0$. This I proceed to show.*

* If $\phi = \| a_{ik} \|$, $(\phi - \lambda)\, e_1 = (a_{11} - \lambda)\, e_1 + a_{21} e_2 + \dots$ and if we write down the corresponding expressions for $(\phi - \lambda)\, e_2$, &c., and use the theorem given at the end of the introduction, we shall get (B) in the form $f\lambda\, [e_1 - e_n]$, and it will be seen that $f\lambda = 0$, the well-known determinant equation giving the latent roots.

The whole investigation depends on the fact that, since we might have expressed x in terms of any n asyzygetic points, we can substitute any n asyzygetic points for the e_i, in (B).

Let λ_1 be any root of (B), let $(\phi-\lambda_1)\,e_i = e_i'$; then we have, by (B),

$$[e_1' e_2' \dots e_n'] = 0.$$

It follows, from this, that we have at least one linear relation connecting the e_i'; but there may be more. Let there be r relations,

$$\sum_j A_{ij}\, e_j' = 0 \quad (i = 1, 2 \dots r) \quad \dots\dots\dots\dots (\mathrm{C}),$$

where $$e_j' = (\phi-\lambda_1)\, e_j.$$

Let $$\Sigma A_{ij}\, e_j = a_i.$$

Since the r relations (c) are asyzygetic, by hypothesis, it follows that the r points a are asyzygetic, for, if they were not, and we had $\Sigma\mu_i a_i = 0$, we should, by operating with $\phi-\lambda_1$, get a relation connecting the equations (C).

It follows that we can substitute the points a for r of the e: suppose we substitute them for $e_1 \dots e_r$; then (B) becomes

$$[\bar{a}_1 \bar{a}_2 \dots \bar{a}_r \bar{e}_{r+1} \dots \bar{e}_n] = 0,$$

if $$\bar{a}_i = (\phi-\lambda)\, a_i.$$

But (C) gives $$(\phi-\lambda_1)\, a_i = 0,$$

or $$\phi a_i = \lambda_1 a_i,$$

and therefore $$(\phi-\lambda)\, a_i = (\lambda_1-\lambda)\, a_i,$$

and (B) becomes $$(\lambda_1-\lambda)^r [a_1 a_2 \dots a_r \bar{e}_{r+1} \dots \bar{e}_n] = 0 \dots\dots\dots\dots\dots\dots (\mathrm{B}).$$

Therefore, if there are asyzygetic relations (C), there are r points a, such that $(\phi-\lambda_1)\, a_i = 0$, and λ_1 is an r-tuple root, at least, of (B). But the multiplicity of λ_1 may be greater than r; if it is, we must have

$$[a_1 a_2 \dots a_r e_{r+1}' \dots e_n'] = 0,$$

$$e_i' = (\phi-\lambda_1)\, e_i.$$

Suppose, as before, that there are s asyzygetic relations

$$\sum_1^r B_{ij}\, a_j - \sum_{r+1}^n B_{ij}\, e_j' = 0 \quad (i = 1 \dots s) \quad \dots\dots\dots\dots (\mathrm{C}').$$

Then all the coefficients $B_{i(r+1)} \dots B_{in}$ cannot vanish, since the a are asyzygetic, and we can take

$$b_i = \sum_{r+1}^{n} B_{ij} e_j \quad (i = 1 \dots s),$$

and substitute the s points b in place of, say, $e_{r+1} \dots e_{r+s}$. We have now to see what (B) becomes. In the first place, we get

$$(\lambda_1 - \lambda)^r [a_1 \dots a_r \bar{b}_1 \dots \bar{b}_s \bar{e}_{r+s+1} \dots \bar{e}_n] = 0 \dots\dots\dots\dots (\mathrm{B}'),$$

if $$\bar{b}_i = (\phi - \lambda)\, b_i.$$

Now (C′) gives $$\sum B_{ij} a_j - (\phi - \lambda_1)\, b_i = 0,$$

for $$\sum B_{ij} e'_j = (\phi - \lambda_1)\, b_i.$$

Therefore $$\phi b_i = \lambda_1 b_i + \sum B_{ij} a_j.$$

Therefore $$(\phi - \lambda)\, b_i = (\lambda_1 - \lambda)\, b_i + \sum B_{ij} a_j.$$

Therefore $$[a_1 \dots a_r \bar{b}_1 \dots \bar{b}_s] = [a_1 \dots a_r] \prod_{i=1}^{i=s} [(\lambda_1 - \lambda)\, b_i + \sum B_{ij} a_j]$$

$$= [a_1 \dots a_r]\, \Pi\, (\lambda_1 - \lambda)\, b_i{}^*$$

$$= (\lambda_1 - \lambda)^s [a_1 \dots a_r b_1 \dots b_s],$$

and (B′) becomes

$$(\lambda_1 - \lambda)^{r+s} [a_1 \dots a_r b_1 \dots b_s \bar{e}_{r+s+1} \dots \bar{e}_n] = 0 \dots\dots\dots\dots (\mathrm{B}'').$$

It is obvious how we might go on if the multiplicity of λ_1 were greater than $r+s$; we should get t points c, such that

$$(\phi - \lambda_1)\, c_i = \sum c_{ij} a_j + \sum c'_{ij} b_j,$$

and then (B″) would become

$$(\lambda_1 - \lambda)^{r+s+t} [a_1 \dots a_r b_1 \dots b_s c_1 \dots c_s \bar{e}_{r+s+t+1} \dots \bar{e}_n] = 0 \dots\dots (\mathrm{B}''').$$

We can now enunciate the following theorem:—The equation

$$\Pi\, (\phi - \lambda)\, e_i = 0$$

[the left-hand side of which is called the latent function of ϕ] has n roots [called the latent roots of ϕ]; to a latent root (λ) of multiplicity a appertain a points; these points group themselves into sets, such that, if we call the points of the first set a_i, those of the second set b_i, and so on, we have

$$\phi = \frac{\lambda a_i, \ \lambda b_i + A_i, \ \lambda c_i + A'_i + B_i \dots}{a_i, \quad b_i, \quad c_i},$$

where A_i, B_i, &c. denote syzygies of the a_i, b_i, &c.

* *Vide* p. 69, l. 8.

Now it is obvious that the A, &c. are asyzygetic: this follows from the way in which they were determined. It follows that their number cannot be greater than that of the a; and obviously $\phi A_i = \lambda A_i$. We can, therefore, substitute the A_i for an equal number of the a, and then we get

$$\phi = \frac{\lambda a_i,\ \lambda b_i + a_i,\ \lambda c_i + A_i + B_i}{a_i,\quad b_i,\quad c_i},$$

where obviously A_i is not the same as before. But now we can substitute $A_i + B_i$ for an equal number of the b. Let $B_i = \Sigma B_{ij} b_j$; then

$$\phi\,(A_i + B_i) = \lambda\,(A_i + B_i) + \Sigma B_{ij}\, a_j.$$

Therefore we must substitute $\Sigma B_{ij} a_j$ for a_i, and then we get

$$\phi = \frac{\lambda a_i,\ \lambda b_i + a_i,\ \lambda c_i + b_i}{a_i,\quad b_i,\quad c_i},$$

and it is obvious how we may proceed.*

The points a_i, b_i, &c. are called the latent points of ϕ appertaining to the latent root λ.

The number of groups a_i, b_i, c_i that we get for any latent root depends on the coefficients of the latent function. There are two cases in which the theory of the latent points is particularly simple:—(1) the case in which no latent root is repeated, so that, for a latent point a_i appertaining to a root λ_i, we have

$$\phi a_i = \lambda_i a_i;$$

and (2) the case in which a latent root is repeated, but all its latent points are a's, so that, again, for all latent points a_{ij} appertaining to λ_i,

$$\phi a_{ij} = \lambda_i a_{ij}.$$

7. We have
$$(\phi - \lambda)\, a_i = 0,$$
$$(\phi - \lambda)\, b_i = a_i,$$
and therefore
$$(\phi - \lambda)^2 b_i = (\phi - \lambda)\, a_i = 0;$$
similarly
$$(\phi - \lambda)^s x_i = 0,$$
if x_i is a latent point in the s^{th} group.

Therefore, if there are s groups of latent points appertaining to λ,

$$(\phi - \lambda)^s e_i = 0,$$

where e_i is any latent point appertaining to λ.

* Jordan, "Traité des Substitutions," 125.

Therefore, if the latent roots are $\lambda_1, \lambda_2 \dots \lambda_r$, and if there are $s_1 \dots s_r$ groups of latent points appertaining to them, then, for any latent point e_i,

$$(\phi-\lambda_1)^{s_1}(\phi-\lambda_2)^{s_2}\dots(\phi-\lambda_r)^{s_r}e_i = 0.$$

But the n points e_i are asyzygetic; therefore, for all points x,

$$(\phi-\lambda_1)^{s_1}(\phi-\lambda_2)^{s_2}\dots(\phi-\lambda_r)^{s_r}x = 0,$$

that is, $$(\phi-\lambda_1)^{s_1}(\phi-\lambda_2)^{s_2}\dots(\phi-\lambda_r)^{s_r} = 0.$$

This is the *identical equation.*

If the roots are all unequal, $r = n$, $s_1 = s_2 = \dots = s_n = 1$, and the equation is

$$(\phi-\lambda_1)(\phi-\lambda_2)\dots(\phi-\lambda_n) = 0.$$

8. We have $$\phi a_i = \lambda a_i,$$

and therefore $$\phi^m a_i = \lambda^m a_i,$$

and generally, if f is any function, not involving matrices other than ϕ,

$$f(\phi)\,a_i = f(\lambda)\,a_i.$$

We have
$$\begin{aligned}\phi b_i &= \lambda b_i + a_i,\\ \phi^2 b_i &= \lambda\phi b_i + \phi a_i\\ &= \lambda(\lambda b_i + a_i) + \lambda a_i\\ &= \lambda^2 b_i + 2\lambda a_i,\end{aligned}$$

and generally $$f(\phi)\,b_i = f(\lambda)\,b_i + f'(\lambda)\,a_i.$$

In the same way, if x_i is in the s^{th} group,

$$f(\phi)\,x_i = f(\lambda)\,x_i + \dots + f^{(s-1)}(\lambda)\,a_i.$$

9. Now let $s_1 = s_2 = \dots = s_r = 1$: then we have, if

$$(\lambda-\lambda_1)(\lambda-\lambda_2)\dots(\lambda-\lambda_r) = \chi\lambda,$$

$$f(\phi) = \Sigma\,\frac{\chi(\phi)}{(\phi-\lambda_i)}\cdot\frac{f\lambda_i}{\chi'\lambda_i} \quad \dots\dots\dots\dots\dots\dots\dots\dots(A),$$

f being any function not involving any matrix other than ϕ.

The function $$\frac{\chi(\phi)}{\phi-\lambda_i}$$

simply means

$$(\phi-\lambda_1)(\phi-\lambda_2)\dots(\phi-\lambda_{i-1})(\phi-\lambda_{i+1})\dots(\phi-\lambda_r).$$

To prove (A), I denote the right-hand side of it by $F(\phi)$, and I show that $f(\phi)\,e_j = F(\phi)\,e_j$, e_j being any latent point. We have, if e_j appertains to λ_j,

$$\frac{\chi(\phi)}{(\phi-\lambda_i)}\,e_j = \frac{\chi\lambda_j}{\lambda_j-\lambda_i}\,e_j$$

$$= 0 \quad \text{if } i \gtrless j$$

$$= \chi'\lambda \quad \text{if } i = j.$$

Therefore $$F(\phi)\,e_j = f(\lambda_j)\,e_j$$

$$= f(\phi)\,e_j,$$

and, therefore, generally $$f(\phi) = F(\phi)^*.$$

If the s are not all equal to unity, the formula is much more complicated.

10. If we have r asyzygetic points e_i, and no more, such that $\phi e_i = 0$, ϕ is said to be of *nullity* r. It follows, from what was proved in (6), that in this case (since $\phi e_i = 0\,.\,e_j$) r, at least, of the latent roots must vanish. But more than r may vanish; and, accordingly, if s latent roots vanish, ϕ is said to be of *vacuity* s. We see that the nullity of a matrix cannot be greater than its vacuity, but may be less. A point x, such that $\phi x = 0$, is called a *null-point* of ϕ: the *r-point* determined by the r asyzygetic null-points of a matrix of nullity r is called the *null-space* of the matrix.

If $e_1 \dots e_r$ are r asyzygetic null-points of a matrix of nullity r, it is obvious that $\Sigma\lambda_i e_i$ is also a null-point; that is, every point in the null-space of a matrix is a null-point. Moreover, it is easy to see that every null-point must be in the null-space; for, if there were a null-point e_{r+1} not in $[e_1 e_2 \dots e_r]$, we should have $(r+1)$ asyzygetic null-points, and the nullity of the matrix would be $r+1$.

Applying what was proved in (6) above, to the case $\lambda = 0$, we see that we get $$\phi = \frac{0 \;\; a_i \;\; b_i \dots}{a_i \;\; b_i \;\; c_i \dots}.$$

We see that, if there are s groups $a, b, c \dots$ and $\alpha, \beta, \gamma \dots$ points, $a_i, b_i, c_i \dots$, the nullity of ϕ is α, and its vacuity $\alpha+\beta+\gamma \dots$; the nullity of ϕ^2 is $\alpha+\beta$ (since $\phi^2 b = 0$, $\phi^2 a = 0$), and its vacuity $\alpha+\beta+\gamma\dots$; the nullity of ϕ^s is $(\alpha+\beta+\gamma\dots)$, and its vacuity $\alpha+\beta+\gamma\dots$ Therefore the nullity of ϕ^s is equal to its vacuity: and, if we apply this to

* (A) is Professor Sylvester's "interpolation formula," giving the standard form to which all functions of a matrix can be reduced: it appears above in what is, I think, a more general form than Professor Sylvester's.

the matrix $\phi-\lambda_i$, where λ_i is a latent root of ϕ having s_i groups, we see that s_i can be defined as the order of the lowest power of $(\phi-\lambda_i)$ of which the nullity is equal to its vacuity.

Consider the equation $y=\phi x$. Given x, this determines y uniquely; given y, x is not determinate unless the nullity of ϕ is zero: for, if we have $y=\phi x=\phi x'$, we have

$$\phi(x-x')=0.$$

Therefore $(x-x')$ must be a null-point of ϕ; and, if ϕ is of nullity r, the solution of the equation $y=\phi x$ contains r arbitrary constants: if x is any solution, the general solution is

$$x+\sum_1^r \lambda_i e_i,$$

the e being r asyzygetic null-points, and the λ arbitrary scalars.

11. Now, take as points of reference r asyzygetic null-points, $e_1 \dots e_r$, and $n-r$ points not in the null-space of ϕ; then, if

$$x=\sum_1^n x_i e_i,$$

$$\phi x=\sum_1^r x_i\phi e_i+\sum_{r+1}^n x_i\phi e_i,$$

$$=\sum_{r+1}^n x_i\phi e_i,$$

since $\phi e_1=\phi e_2=\dots=\phi e_r=0$: therefore the point ϕx is in a certain $(n-r)$ point; viz., the $(n-r)$ point

$$\Pi\equiv[\phi e_{r+1}.\phi e_{r+2}\dots\phi e_n].$$

This product does not vanish; for, if it did, there would be a relation

$$\sum_{r+1}^n c_i\phi e_i=0,$$

that is, $$\phi\Sigma c_i e_i=0.$$

Therefore there would be a null-point of ϕ in $[e_{r+1}\dots e_n]$, which is contrary to the hypothesis.

The $(n-r)$ point Π is called the *latent* space of ϕ: it is obvious that it contains all the latent points for which λ does not vanish.

If the vacuity of ϕ is greater than r, it follows, from what was proved above, that the latent space of ϕ will contain some or all of its null-points.

12. Let ϕ, χ be two matrices of nullities r, s respectively: it is required to find the nullity of $\phi\chi$. I shall show that, if the null-space

of ϕ intersects the latent space of χ in a t-point, the nullity of $\phi\chi$ is $s+t$.*

I take as points of reference, $e_1 \dots e_{n-s}$, being $(n-s)$ asyzygetic points, not situate in the null-space of χ, and $e_{n-s+1} \dots e_n$ asyzygetic null-points of χ; let the null-space of ϕ cut the latent space of χ in the t-point $[E_1 \dots E_t]$, where the E are, of course, supposed to be asyzygetic; lastly, let $\chi e_i = e_i'$. Let $A = \Sigma A_i e_i$ be any point: then

$$\chi A = \sum_1^n A_i \chi e_i$$

$$= \sum_1^{n-s} A_i e_i',$$

since $e_{n-s+1}' = \dots = e_n' = 0$; therefore

$$\phi\chi A = \sum_1^{n-s} A_i \phi e_i'.$$

But we can select from $e_1' \dots e_{n-s}'$, $n-s-t$ points, asyzygetic with the E, and then we have

$$e_i' = \sum_{k=1}^{t} \beta_{ik} E_k + \sum_{j=t+1}^{n-s} a_{ij} e_j' \; (i = 1 \dots t) \quad \dots\dots\dots\dots\dots\dots (\text{X}).$$

Therefore, since $\phi E_k = 0$ (since the E are in the null-space of ϕ),

$$\phi e_i' = \Sigma\, a_{ij} \phi\, e_j',$$

and

$$\phi\chi A = \sum_{j=t+1}^{n-s} \phi e_j' \, (A_j + \sum_{i=1}^{t} a_{ij} A_i).$$

Therefore all points ϕ, χ, A are in the $(n-s-t)$ point $[\phi e_{t+1}' \dots \phi e_{n-s}]$; and, to show that this is actually the latent space of $\phi\chi$, we have only to show that these $(n-s-t)$ points are asyzygetic: but if they were not asyzygetic, and we had

$$0 = \sum_1^{n-s-t} c_i \phi\, e_{t+i}'$$

$$= \phi \Sigma c_i e_{t+i}',$$

we must have

$$\Sigma c_i e_{t+i}' = \Sigma \lambda_j E_j,$$

which is contrary to the supposition on which e_{t+1}', &c. were selected. Therefore the latent space of $\phi\chi$ is an $(n-s-t)$ point, and therefore its nullity is $(s+t)$; and it can be shown without difficulty that its null space is the $(s+t)$ point joining the null space of χ to the t-point in its latent space, which χ transforms into $[E_1 \dots E_t]$.

13. In all that follows, I shall assume that, in the notation of (7), $s_1 = s_2 = s_r = 1$.†

* Cf. *Phil. Mag.*, Nov. 1884.

† That is, that all latent points are a's: Case (2) of (6).

Now let it be proposed to find a non-vacuous* matrix ψ such that $\psi\phi = \phi\psi$, ϕ being a given matrix.

Let $e_1 \dots e_n$ be the latent points of ϕ. Let e_i appertain to λ_i; suppose $\lambda_1 = \lambda_2 = \dots = \lambda a_1 = \lambda_1$, &c.; lastly, let

$$\psi e_i = \Sigma a_{ij} e_j.$$

Then
$$\phi\psi e_i = \Sigma a_{ij}\phi e_j$$
$$= \Sigma a_{ij}\lambda_j e_j.$$

But
$$\psi\phi e_i = \psi\lambda_i e_i$$
$$= \lambda_i \Sigma a_{ij} e_j.$$

Therefore, since $\psi\phi = \phi\psi$, and ψ is supposed non-vacuous, we have

$$a_{ij}\lambda_j = a_{ij}\lambda_i.$$

Therefore, unless $\lambda_i = \lambda_j$, $a_{ij} = 0$; and it follows that ψ transforms all latent points appertaining to the same latent root λ_i into points of the same a_i-point, if a_i is the multiplicity of λ_i; we therefore have

$$\psi = \Sigma\psi_i,$$

where ψ_i is a matrix of nullity $n - a_i$, having $[e_1 \dots e_{a_i}]$ as its latent space, and therefore transforming every latent point appertaining to λ_i into another latent point appertaining to λ_i.

If $a_1 = a_2 = \dots = a_r = 1$, we can go further than this, and can assign the form of ψ; for in this case it is obvious that ψ must transform every latent point into itself; that is,

$$\psi = \frac{\Lambda_i e_i}{e_i}.$$

But
$$\frac{\Lambda_i e_i}{e_i} = \Sigma \frac{\chi(\phi)}{\phi - \lambda_i} \cdot \frac{\Lambda_i}{\chi'\lambda},$$

using the same notation as in (9). This can be proved by the method there employed. Therefore we can say that, if $\phi\psi = \psi\phi$, and if ψ is non-vacuous, and the latent roots of ϕ are all different, ψ is a function of ϕ, of order $(n-1)$, and with scalar coefficients; viz., we have

$$\psi = \Sigma \frac{\chi(\phi)}{\phi - \lambda_i} \cdot \frac{\Lambda_i}{\chi'(\lambda_i)},$$

* A matrix is vacuous if its vacuity $\geqq 1$.

where the Λ are scalars, and $\chi\lambda$ is the latent function of ϕ, viz.,

$$\chi\lambda = (\lambda - \lambda_1) \ldots (\lambda - \lambda_n).^*$$

We can apply a similar method to the equation $\phi\psi = k\psi\phi$, k a scalar. We shall find that k is an n^{th} root of unity, and that, if it is a primitive m^{th} root of unity, ψ is equivalent to a substitution operating on the n latent points of ϕ; viz., if $n = mm'$, ψ is the product of m' cyclic substitutions, each cycle being of order m.

14. As a last example of the methods of this paper, I take the solution of the general unilateral equation.

Let the given equation be

$$0 = F(x) = A_0 x^m + A_1 x^{m-1} + \ldots + A_m,$$

where the A are known matrices of order n, and x is an unknown matrix of the same order.

Let $\lambda_1 \ldots \lambda_n$ be the latent roots of x; $e_1 \ldots e_n$ its latent points. Since $F(x) = 0$, we have

$$\begin{aligned} 0 &= F(x)\, e_i \\ &= F(\lambda_i)\, e_i. \end{aligned}$$

Therefore e_i must be a null-point of $F(\lambda_i)$; $F(\lambda_i)$ must be vacuous, and therefore, if we take n points of reference a_j, we must have

$$\prod_{j=1}^{n} [F(\lambda_i)\, a_j] = 0\dagger \quad (i = 1 \ldots n),$$

that is, the latent roots of x must be roots of this equation of order mn, and, if we take any set of n roots, the latent point of x appertaining to a root λ_i is a null-point of $F\lambda_i$, and x is thus completely determined.

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* *Cf.* Clifford's Math. Papers, 339.

† This equation is simply Det $(F\lambda_n) = 0$.

Results from a Theory of Transformation of Elliptic Functions.

By JOHN GRIFFITHS, M.A.

[*Read Nov.* 13*th*, 1884.]

CONTENTS OF PAPER.

General Outline.

The following note is an attempt to further develope the theory in question, which is based on the assumption that a transformation-equation is composed of two or more equations of lower degree. Th

forms hitherto used by me are practically identical with the following, viz.,

1. $$y = \sin(\theta + A + B + C + \ldots),$$

where $\sin\theta = x$; $\cos A = \dfrac{1-(1+\alpha'^2)x^2}{1-\alpha^2x^2}$, $\sin A = \dfrac{2\alpha' x\sqrt{1-x^2}}{1-\alpha^2x^2}$,

$$\cos B = \frac{1-(1+\beta'^2)x^2}{1-\beta^2x^2}, \quad \sin B = \frac{2\beta' x\sqrt{1-x^2}}{1-\beta^2x^2}, \text{ \&c.,}$$

and the coefficients are of the forms

$$\alpha = k\,\mathrm{sn}\,\frac{2K}{n}, \quad \alpha' = \mathrm{dn}\,\frac{2K}{n}, \quad \beta = k\,\mathrm{sn}\,\frac{4K}{n}, \quad \gamma = k\,\mathrm{sn}\,\frac{6K}{n}, \text{ \&c.,}$$

if n be the order of the transformation, and $\frac{1}{2}(n-1)$ the number of coefficients.

2. $$y = \sin(L + A + B + C + \ldots).$$

This is an irrational transformation-equation, wherein

$$\sin L = \frac{(1+k')x\sqrt{1-x^2}}{\sqrt{1-k^2x^2}}, \quad \cos L = \frac{1-(1+k')x^2}{\sqrt{1-k^2x^2}},$$

$$\sin A = \frac{2a'x\sqrt{1-x^2}}{1-a^2x^2}, \qquad \cos A = \frac{1-(1+a'^2)x^2}{1-a^2x^2},$$

$$\sin B = \frac{2b'x\sqrt{1-x^2}}{1-b^2x^2}, \qquad \cos B = \frac{1-(1+b'^2)x^2}{1-b^2x^2}, \text{ \&c.,}$$

$$a \equiv k\,\mathrm{sn}\,\frac{\delta K}{n}, \quad a' \equiv \mathrm{dn}\,\frac{\delta K}{n}; \quad b = k\,\mathrm{sn}\,\frac{2K}{n}, \text{ \&c.;}$$

$2n$ the order of the transformation, and $n-1$ the number of coefficients.

Intimately connected with (2) is an equation (3), viz.,

$$y = \cos(A + B + C + \ldots);$$

i.e., y = rational and integral function of x ÷ ditto of the same degree.

But, besides the forms in question, there is also a remarkable one which may be called the complementary equation; this is

$$iy = \tan(X + X_1 + X_2 + \ldots) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

X, X_1, X_2, &c. being functions of x, k, and $i = \sqrt{-1}$.

The note is divided into Parts I. and II. In the former it is shown

that the formula (4) includes not only Jacobi's rational transformation-equation $y = \frac{x\,(1,\, x^2)^{\frac{1}{2}(n-1)}}{(1,\, x^2)^{\frac{1}{2}(n-1)}}$ (n an odd number), but also others of interest. In Part II. the theory of composition is applied to the transformation of Θ-functions, and a multiplication theorem, not hitherto noticed, is proved.

It is convenient to state here some lemmas used in the sequel.

Lemma 1.—If

$$\tan X = ix, \quad \tan X_1 = 2ia_1x \div (1+a_1^2x^2),$$

$$\tan X_2 = 2ia_2x \div (1+a_2^2x^2), \text{ \&c.}, \quad i = \sqrt{-1},$$

then
$$i \tan (X+X_1+\ldots+X_m) = \frac{x\,(1,\, x^2)^m}{(1,\, x^2)^m},$$

where $(1, x^2)^m$ denotes a rational and integral function of x^2 of the degree m.

Here
$$\sin X = ix \div \sqrt{1-x^2}, \quad \cos X = 1 \div \sqrt{1-x^2};$$

$$\sin X_1 = 2ia_1x \div 1-a_1^2x^2, \quad \cos X_1 = 1+a_1^2x^2 \div 1-a_1^2x^2, \text{ \&c.},$$

and, therefore,

$$\sqrt{1-x^2} \sin (X+X_1) = ix\,(1+2a_1+a_1^2x^2) \div 1-a_1^2x^2,$$

$$\sqrt{1-x^2} \cos (X+X_1) = 1+(a_1^2+2a_1)\,x^2 \div 1-a_1^2x^2;$$

similarly

$$\sqrt{1-x^2} \sin (X+X_1+X_2) = ix\,(1,\, x^2)^2 \div (1-a_1^2x^2)(1-a_2^2x^2),$$

$$\sqrt{1-x^2} \cos (X+X_1+X_2) = (1,\, x^2)^2 \div (1-a_1^2x^2)(1-a_2^2x^2).$$

Generally, by a process of induction, it is inferred that

$$\sqrt{1-x^2} \sin (X+X_1+\ldots+X_m) = ix\,(1,\, x^2)^m \div (1-a_1^2x^2) \ldots (1-a_m^2x^2),$$

$$\sqrt{1-x^2} \cos (X+X_1+\ldots+X_m) = (1,\, x^2)^m \div (1-a_1^2x^2) \ldots (1-a_m^2x^2).$$

Lemma 2.—If

$$\tan X = \frac{i\,(1+k)\,x}{1+kx^2}, \quad \tan X_1 = \frac{2ib_1x}{1+b_1^2x^2}, \quad \tan X_2 = \frac{2ib_2x}{1+b_2^2x^2}, \text{ \&c.},$$

then
$$i \tan (X+X_1+\ldots+X_m) = i\,\frac{x\,(1,\, x^2)^m}{(1,\, x^2)^{m+1}}.$$

Here $$\sin X = i\,(1+k)\,x \div \sqrt{1-x^2\,.\,1-k^2x^2},$$
$$\cos X = 1+kx^2 \div \sqrt{1-x^2\,.\,1-k^2x^2},$$
$$\sin X_1 = 2ib_1x \div 1-b_1^2x^2, \quad \cos X_1 = 1+b_1^2x^2 \div 1-b_1^2x^2, \ \&c.$$
Hence it is inferred that
$$\sqrt{1-x^2\,.\,1-k^2x^2}\,\sin(X+X_1+\ldots+X_m) = ix\,(1,\,x^2)^m \div,$$
$$\sqrt{1-x^2\,.\,1-k^2x^2}\,\cos(X+X_1+\ldots+X_m) = (1,\,x^2)^{m+1} \div,$$
$$\text{denominator} = (1-a_1^2x^2)\,(1-a_2^2x^2)\,\ldots\,(1-a_m^2x^2).$$

Lemma 3.—If
$$\tan X = i, \quad \tan X_1 = 2ia_1 \div (1+a_1^2), \quad \tan X_2 = 2ia_2 \div (1+a_2^2), \ \&c.,$$
then $$\tan(X+X_1+\ldots+X_m) = i.$$

Here $$\tan(X+X_1) = i\left(1+\frac{2a_1}{1+a_1^2}\right) \div 1+\frac{2a_1}{1+a^2} = i;$$
similarly, $$\tan(X+X_1+X_2) = i,$$
and so, generally, $$\tan(X+X_1+\ldots+X_m) = i.$$

Lemma 4.—If $$iy = \tan(X+X_1+\ldots)$$
and $$iM\sqrt{\frac{1-\lambda^2y^2}{1-y^2}} = \sqrt{1-x^2\,.\,1-k^2x^2}\ \Sigma\,\frac{dX}{dx}$$
(where M is a constant, and X, X_1 ... are functions of x, k, $i = \sqrt{-1}$) be two forms of the same integral relation between y and x, then this relation must give rise to the differential equation
$$\frac{dy}{\sqrt{1-y^2\,.\,1-\lambda^2y^2}} = \frac{M\,dx}{\sqrt{1-x^2\,.\,1-k^2x^2}}.$$

The lemma may be proved without difficulty; we have, by differentiation,
$$idy = \sec^2(X+X_1+\ldots)\,\Sigma\,\frac{dX}{dx}\,dx, \quad i.e., \ \frac{i\,dy}{1-y^2} = \Sigma\,\frac{dX}{dx}\,dx,$$
or $$\frac{dy}{\sqrt{1-y^2\,.\,1-\lambda^2y^2}} = \frac{M\,dx}{\sqrt{1-x^2\,.\,1-k^2x^2}},$$
if $$iM\sqrt{\frac{1-\lambda^2y^2}{1-y^2}} = \sqrt{1-x^2\,.\,1-k^2x^2}\ \Sigma\,\frac{dX}{dx}.$$

This may, at first sight, not appear to be of much practical use in the problem of transformation, but the sequel will show that some forms of X, X_1, &c. satisfy the required conditions.

PART I.

Section 1.—Jacobi's rational transformation-equation

$$y = \frac{x\,(1,\ x^2)^{\frac{1}{2}(n-1)}}{(1,\ x^2)^{\frac{1}{2}(n-1)}}\ (n \text{ odd prime, say}).$$

This equation is given by the formula

$$iy = \tan\,(X + X_1 + \ldots + X_{\frac{1}{2}(n-1)}),$$

where
$$\tan X = ix,$$

$$\tan X_1 = 2ia_1 x \div (1 + a_1^2 x^2)\ \ldots,$$

$$\tan X_{\frac{1}{2}(n-1)} = 2ia_{\frac{1}{2}(n-1)}x \div (1 + a_{\frac{1}{2}(n-1)}^2 x^2).$$

Here, by Lemma 1, we have

$$y = x\,(1,\ x^2)^{\frac{1}{2}(n-1)} \div (1,\ x^2)^{\frac{1}{2}(n-1)},$$

$$\sqrt{1-y^2} = \sec\,(X + X_1 + \ldots + X_{\frac{1}{2}(n-1)})$$
$$= \sqrt{1-x^2}\,(1-a_1^2x^2)\ \ldots\ (1-a_{\frac{1}{2}(n-1)}^2x^2) \div (1,\ x^2)^{\frac{1}{2}(n-1)},$$

and also $\dfrac{dy}{\sqrt{1-y^2\,.\,1-\lambda^2y^2}} = M\dfrac{dx}{\sqrt{1-x^2\,.\,1-k^2x^2}}$ (by Lemma 4),

provided that

$$iM\sqrt{\frac{1-\lambda^2y^2}{1-y^2}} = \sqrt{1-x^2\,.\,1-k^2x^2}\ \Sigma\frac{dX}{dx};$$

i.e., since $\dfrac{dX}{dx} = i \div 1-x^2$, $\dfrac{dX_1}{dx} = 2ia_1 \div (1-a_1^2x^2)$, &c.,

the condition is reduced to

$$M\sqrt{1-\lambda^2y^2} = \sqrt{1-k^2x^2} \times \text{rational function of } x^2.$$

This, in effect, leads to Jacobi's method (see Prof. Cayley's "Elliptic Functions," Chap. 7).

As a particular case, I note that Jacobi's *first* transformation may be written in either of the forms

$$y = \sin\,(\theta + A + B + \ldots)$$

[see General Outline], or

$$(-)^{\frac{1}{2}(n-1)}\, iy = \tan\left(X - X_1 + X_2 - \ldots \mp X_{\frac{1}{2}(n-3)} \pm X_{\frac{1}{2}(n-1)}\right),$$

where $\tan X = ix$, $\tan X_m = 2ia_m x \div (1 + a_m^2 x^2)$.

As regards the coefficients $a_1 \ldots a_{\frac{1}{2}(n-1)}$, the equation

$$\sqrt{1-y^2} = \sqrt{1-x^2}\,(1-a_1^2x^2)\ldots(1-a_{\frac{1}{2}(n-1)}^2x^2) \div (1,\, x^2)^{\frac{1}{2}(n-1)}$$

enables us at once to identify them, and write

$$a_1 = \frac{\operatorname{dn}\frac{2K}{n}}{\operatorname{cn}\frac{2K}{n}}, \quad \ldots \quad a_{\frac{1}{2}(n-1)} = \frac{\operatorname{dn}\frac{n-1}{n}K}{\operatorname{cn}\frac{n-1}{n}K},$$

or, generally,

$$a_s = \frac{\operatorname{dn}\frac{2sK}{n}}{\operatorname{cn}\frac{2sK}{n}}.$$

The two forms in question give some interesting relations between the coefficients; for example,

$$(-)^{\frac{1}{2}(n-1)}\left\{1 + 2\Sigma \operatorname{dn}\frac{2sK}{n}\right\} = (-)^{\frac{1}{2}(n-1)}M = 1 + 2\Sigma(-)^s \operatorname{dn}\left(\frac{2siK}{n}, k'\right),$$

where s is an integer from 1 to $\frac{1}{2}(n-1)$, and n an odd prime.

In particular, if $k=0$, then

$$1 + 2\Sigma(-)^s \sec\frac{s\pi}{n} = (-)^{\frac{1}{2}(n-1)}n.$$

Again,

$$2k^2\Sigma \operatorname{sn}^2\frac{2sK}{n} + 2k'^2\Sigma \operatorname{sn}^2\left(\frac{2siK}{n},\, k'\right) + \left\{1 + 2\Sigma \operatorname{dn}\frac{2sK}{n}\right\}^2 = n.$$

Section 2.—Deduction of a rational transformation-equation from

$$y = \sin(L + A + B + \ldots).$$

(See General Outline.)

Lemma 5.—Given a transformation-equation $y = f(x, k)$, and the complementary one $y' = \phi(x', k')$; then, if the former be considered as a primary transformation, there is a secondary form $z = \phi(x, k)$, which is the exact analogue of $y' = \phi'(x', k')$, and also a secondary complementary transformation $z' = f(x', k')$, which is the analogue of $y = f(x, k)$.

For example, from Jacobi's first transformation and its complementary form, we deduce the modular relations

$$n\Lambda = M.K \text{ and } \Lambda' = M.K';$$

hence there is a second transformation in which the modular equations are

$$\Lambda_1 = M_1.K \text{ and } n\Lambda_1' = M_1.K'.$$

Here $\Lambda_1 = M_1.K$ is derived from the primary complementary relation $\Lambda' = M.K'$ by changing λ', k', M into λ_1, k, M_1.

Similarly $n\Lambda_1' = M_1.K'$ is derived from $n\Lambda = M.K$.

1°. In accordance with this lemma, it is necessary to consider the complementary form of

$$y = \sin(L+A+B+\ldots),$$

where
$$\cos L = \frac{1-(1+k')\,x^2}{\sqrt{1-k^2x^2}}, \quad \sin L = \frac{(1+k')\,x\sqrt{1-x^2}}{\sqrt{1-k^2x^2}};$$

$$\cos A = \frac{1-(1+a'^2)\,x^2}{1-a^2x^2}, \text{ \&c.}$$

(See General Outline.)

Here
$$\sqrt{1-y^2} = \cos(L+A+B+\ldots)$$
$$= \text{rational function of } x \div \sqrt{1-k^2x^2},$$

and
$$M\sqrt{1-\lambda^2y^2} = \sqrt{1-k^2x^2}\left\{1+\frac{k'}{1-k^2x^2}+2\Sigma\frac{a'}{1-a^2x^2}\right\}.$$

This last equation is, in fact, the condition that the integral form

$$y = \sin(L+A+B+\ldots)$$

shall give the differential equation

$$\frac{dy}{\sqrt{1-y^2\,.\,1-\lambda^2y^2}} = M\frac{dx}{\sqrt{1-x^2\,.\,1-k^2x^2}}.$$

Let
$$\frac{y}{\sqrt{1-y^2}} = iy', \quad \frac{x}{\sqrt{1-x^2}} = ix',$$

then we have
$$\frac{dy'}{\sqrt{1-y'^2\,.\,1-\lambda'^2y'^2}} = M\frac{dx'}{\sqrt{1-x'^2\,.\,1-k'^2x'^2}},$$

if
$$\lambda^2+\lambda'^2 = 1 = k^2+k'^2.$$

To find the integral relation between y' and x', write

$$\frac{y}{\sqrt{1-y^2}} = \tan(L+A+B+\ldots),$$

where
$$\tan L = \frac{(1+k')\,x\sqrt{1-x^2}}{1-(1+k')\,x^2} = \frac{i(1+k')\,x'}{1+k'x'^2},$$

$$\tan A = \frac{2a'x\sqrt{1-x^2}}{1-(1+a'^2)\,x^2} = \frac{2ia'x'}{1+a'^2x'^2},$$

$$\tan B = \frac{2b'x\sqrt{1-x^2}}{1-(1+b'^2)\,x^2} = \frac{2ib'x'}{1+b'^2x'^2}, \text{ \&c.}$$

Hence the required relation is

$$iy' = \tan(X'+Y'+Z'+\ldots),$$

where
$$\tan X' = \frac{i(1+k')\,x'}{1+k'x'^2},$$

$$\tan Y' = \frac{2ia'x'}{1+a'^2x'^2}, \quad \tan Z' = \frac{2ib'x'}{1+b'^2x'^2}, \text{ \&c.}$$

This gives, as we have seen above,

$$\frac{dy'}{\sqrt{1-y'^2\,.\,1-\lambda'^2y'^2}} = M\frac{dx'}{\sqrt{1-x'^2\,.\,1-k'^2x'^2}}.$$

Change y', x', λ', k', M into z, x, γ, k, N, and the coefficients a', b', &c. into a_1, a_2, &c., then

$$\frac{dz}{\sqrt{1-z^2\,.\,1-\gamma^2z^2}} = N\frac{dx}{\sqrt{1-x^2\,.\,1-k^2x^2}},$$

if
$$iz = \tan(X+X_1+X_2+\ldots),$$

where
$$\tan X = \frac{i(1+k)\,x}{1+kx^2},$$

$$\tan X_1 = \frac{2ia_1x}{1+a_1^2x^2}, \quad \tan X_2 = \frac{2ia_2x}{1+a_2^2x^2}, \text{ \&c.,}$$

and the coefficients are of the type-form

$$a_s = \operatorname{dn}\left(\frac{sK'}{n}, k'\right),$$

$2n$ being the order of the transformation and s an integer from 1 to $n-1$.

In other words, given a primary transformation-equation (irra-

tional), viz.,
$$y = \sin(L+A+\ldots),$$
there is a secondary equation (rational) of the form
$$iz = \tan(X+X_1+\ldots).$$

Here (see Lemma 2),
$$\sqrt{1-z^2} = \sqrt{1-x^2 \,.\, 1-k^2x^2} \times \text{rational function of } x,$$
and consequently, by Lemma 4,
$$\sqrt{1-\gamma^2 z^2} = \text{rational function of } x.$$

2°. The modular relations for the primary and secondary forms in question are
$$2n\Lambda = M.K, \quad \Lambda' = M.K'; \quad \Gamma = N.K, \quad 2n\Gamma' = N.K$$
(order of transformation $= 2n$).

These may be proved without difficulty.

Let $\quad y = \sin\phi, \quad x = \sin\theta, \quad \sin L = \dfrac{(1+k')\sin\theta\cos\theta}{\sqrt{1-k^2\sin^2\theta}},$
$$\sin A = \frac{2a'\sin\theta\cos\theta}{1-a^2\sin^2\theta}, \quad \cos A = \frac{1-(1+a'^2)\sin^2\theta}{1-a^2\sin^2\theta}, \text{ \&c.,}$$
then
$$\phi = L+A+B+\ldots,$$
giving
$$\frac{d\phi}{\sqrt{1-\lambda^2\sin^2\phi}} = M\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

Again, let
$$\theta = \frac{\pi}{2},$$
i.e.,
$$L = 2\,.\,\frac{\pi}{2} = A = B, \text{ \&c.,} \quad \phi = 2n\,.\,\frac{\pi}{2},$$
then
$$\int_0^{2n\frac{1}{2}\pi}\frac{d\phi}{\sqrt{1-\lambda^2\sin^2\phi}} = M\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}},$$
or
$$2n\Lambda = M.K.$$

In the complementary form
$$iy' = \tan(X'+Y'+\ldots),$$
which gives $\quad \dfrac{dy'}{\sqrt{1-y'^2 \,.\, 1-\lambda'^2y'^2}} = M\dfrac{dx'}{\sqrt{1-x'^2 \,.\, 1-k'^2x'^2}},$

let $x' = 1$, and, consequently,

$$\tan X' = i, \quad \tan Y' = \frac{2ia'}{1+a'^2}, \ \&c.;$$

then (by Lemma 3)

$$y' = 1, \ \textit{i.e.}, \ \Lambda' = M.K'.$$

Hence, by the principle of duality,

$$\Gamma = N.K \quad \text{and} \quad 2n\Gamma' = N.K'.$$

3°. Multiplication by $2n$.

The two transformations considered above—viz., an irrational, or primary, and a rational, or secondary, transformation—lead to complete multiplication by $2n$ in exactly the same way as do Jacobi's first and second transformations. (Compare pp. 174, 175 of Prof. Cayley's "Elliptic Functions.")

Section 3.—Composition of linear forms. Application of the formula $iy = \tan(X + X_1 + \ldots)$ to the quadric transformations of Abel, &c.

Taking the two linear forms

$$y = \frac{ax+b}{cx+d}, \quad z = \frac{a'x+b'}{c'x+d'},$$

we may say that a quadric transformation is

$$u = \frac{y+z}{1+yz},$$

or, what is a more convenient formula to work with,

$$iu = \tan(X + X'),$$

where $$\tan X = i\,\frac{ax+b}{cx+d}, \quad \tan X' = i\,\frac{a'x+b'}{c'x+d'}.$$

This, when written at length, gives

$$u = \frac{a_0 + a_1 x + a_2 x^2}{b_0 + b_1 x + b_2 x^2},$$

where a_0, a_1, ... are functions of a, b, &c.

In order that $iu = \tan(X + X')$ may lead to the differential equation

$$\frac{du}{\sqrt{1-u^2 \,.\, 1-\lambda^2 u^2}} = M\,\frac{dx}{\sqrt{1-x^2 \,.\, 1-k^2 x^2}},$$

we must have, by the present method, since

$$\frac{dX}{dx} = i\,\frac{ad-bc}{(cx+d)^2-(ax+b)^2},$$

$$M\sqrt{\frac{1-\lambda^2u^2}{1-u^2}} = \sqrt{1-x^2\,.\,1-k^2x^2}\left\{\frac{ad-bc}{R}+\frac{a'd'-b'c'}{R'}\right\},$$

where $R = (cx+d)^2-(ax+b)^2$, $R' = (c'x+d')^2-(a'x+b')^2$.

Also there is no difficulty in finding

$$\sqrt{1-u^2} = \frac{\sqrt{RR'}}{(ax+b)(a'x+b')+(cx+d)(c'x+d')}.$$

Hence $M\sqrt{1-\lambda^2u^2} = \sqrt{1-x^2\,.\,1-k^2x^2}\,.\,\phi(x) \div \sqrt{RR'}\,f(x)$,

where $$\phi(x) = (ad-bc)\,R'+(a'd'-b'c')\,R,$$

$$f(x) = (ax+b)(a'x+b')+(cx+d)(c'x+d').$$

The different kinds of results will depend upon the different forms of $\sqrt{1-\lambda^2u^2}$ as a function of x. For example, suppose

$$\sqrt{1-\lambda^2u^2} = \text{rational function of } x;$$

then $$RR' \equiv (1-x^2)(1-k^2x^2)$$

to a constant factor *près*. This condition is satisfied by taking

$$R = (ax+1)^2-(x+a)^2 = (1-a^2)(1-x^2),$$

$$R' = (akx-1)^2-(a-kx)^2 = (1-a^2)(1-k^2x^2).$$

Hence $$iu = \tan(X+X'),$$

where $$\tan X = i\,.\,\frac{ax+1}{x+a},$$

$$\tan X' = i\,.\,\frac{akx-1}{-kx+a}, \quad\text{or}\quad u = \frac{(1+k)\,x}{1+kx^2};$$

$$M\,.\,\sqrt{1-\lambda^2u^2} = \frac{(1+k)(1-kx^2)}{1+kx^2}.$$

Let $$x = \frac{1}{\sqrt{k}}, \text{ then } u = \tfrac{1}{2}\,\frac{(1+k)}{\sqrt{k}},$$

and therefore $$\lambda = \frac{2\sqrt{k}}{1+k}, \quad M = 1+k.$$

(See Prof. Cayley's "Elliptic Functions," p. 374.)

I write down a few of the principal forms.

(1) $$\tan X = i\,.\,\frac{-i+\sqrt{k}x}{1-i\sqrt{k}x},\quad \tan X' = i\,.\,\frac{i+\sqrt{k}x}{1+i\sqrt{k}x},$$

$$iu = \tan(X+X') \quad \text{or} \quad u = \frac{2\sqrt{k}\,x}{1+kx^2}.$$

(2) $$\tan X = i\,.\,\frac{x+b}{x+2-b},\quad \tan X' = i\,.\,\frac{x-b}{x-2+b},$$

$$u = \frac{2b-b^2-x^2}{2-2b+b^2-x^2}.$$

(3) $$\tan X = i\,.\,\frac{ax+1}{(2-a)\,x+1},\quad \tan X' = i\,.\,\frac{ax+1}{-(2+a)\,x+1},$$

$$u = \frac{1-a^2x^2}{1-(2-a^2)\,x^2}.$$

(4)
$$\tan X = i\,.\,\frac{x+\dfrac{1}{\beta^2}+\dfrac{1}{2}}{\dfrac{1}{\beta^2}-\dfrac{1}{2}},\qquad \tan X' = i\,.\,\frac{(a^2-1)(a-\beta)\,x+\dfrac{a\beta^2}{2}-\beta}{(a^2-1)(a+\beta)\,x+\dfrac{a\beta^2}{2}+\beta},$$

$$\beta \equiv \text{eighth root of } 16k^2,\quad a^2-1 = \tfrac{1}{4}\beta^4,$$

$$u = \frac{a+\beta}{a-\beta}\,.\,\frac{1+a\beta x+\frac{1}{4}\beta^4x^2}{1-a\beta x+\frac{1}{4}\beta^4x^2}.$$

The above are some of the principal forms of the Abelian quadric transformations. (See Prof. Cayley's "Elliptic Functions," p. 374.)

From the results just obtained and those of Sections 1, 2, it is inferred that the method of composition of linear forms is applicable to the rational transformations of any order which are considered in this note.

For example, Jacobi's quintic equation may be written

$$iy = \tan(X+X_1+X_1'+X_2+X_2'),\quad \text{if} \quad \tan X = ix,$$

$$\tan X_1 = i\,.\,\frac{-i+a_1x}{1-ia_1x},\quad \tan X_1' = i\,.\,\frac{i+a_1x}{1+ia_1x},$$

$$\tan X_2 = i\,.\,\frac{-i+a_2x}{1-ia_2x},\quad \tan X_2' = i\,.\,\frac{i+a_2x}{1+ia_2x}.$$

Generally, $$iu = \tan(X+X'+\ldots),$$

where $$\tan X = i \cdot \frac{ax+b}{cx+d}, \quad \tan X' = i \cdot \frac{a'x+b'}{c'x+d'}, \text{ \&c.},$$

gives rise to the differential equation

$$\frac{du}{\sqrt{1-u^2 \,.\, 1-\lambda^2u^2}} = M \frac{dx}{\sqrt{1-x^2 \,.\, 1-k^2x^2}}$$

if $$M \,.\, \sqrt{\frac{1-\lambda^2u^2}{1-u^2}} = \sqrt{1-x^2 \,.\, 1-k^2x^2}\; \Sigma \frac{ad-bc}{R},$$

where $$\sqrt{1-u^2} = \sqrt{RR'} \ldots \div \text{rational function of } x,$$

and $$R = (cx+d)^2-(ax+b)^2, \text{ \&c.}$$

For example,

$$iu = \tan(X+X'+X'') \quad \text{or} \quad u = \frac{a_0+a_1x+a_2x^2+a_3x^3}{b_0+b_1x+b_2x^2+b_3x^3}$$

includes all the possible rational transformations of order 3. The complete determination of the actual forms would, however, be somewhat difficult.

I append one of them.

If $\tan X = ikx, \quad \tan Y = i \cdot \dfrac{ax-i}{-iax+1}, \quad \tan Z = i \cdot \dfrac{ax+i}{iax+1},$

then $$iu = \tan(X+Y+Z) \quad \text{is} \quad u = \frac{x\,(2a+k+a^2kx^2)}{1+(a^2+2ak)\,x^2}.$$

Also $$\sqrt{1-u^2} = \frac{(1-a^2x^2)\sqrt{1-k^2x^2}}{1+(a^2+2ak)\,x^2},$$

$$M\sqrt{\frac{1-\lambda^2u^2}{1-u^2}} = \sqrt{1-x^2 \,.\, 1-k^2x^2}\left\{\frac{k}{1-k^2x^2}+\frac{2a}{1-a^2x^2}\right\},$$

or $$M\sqrt{1-\lambda^2u^2} = \sqrt{1-x^2}\left\{\frac{2a+k-ak\,\overline{a+2k}\,x^2}{1+(a^2+2ak)\,x^2}\right\}.$$

Let $x=1$, therefore

$$\frac{1}{\lambda} = u = \frac{2a+(1+a^2)\,k}{1+a^2+2ak}.$$

Let $\quad x^2 = \dfrac{2a+k}{ak\,(a+2k)}, \quad$ then $\quad \dfrac{1}{\lambda} = \sqrt{\dfrac{k}{a}}\left(\dfrac{2a+k}{a+2k}\right).$

Hence, if $k = \mu a$, the ultimate results are

$$\frac{1}{\lambda^2} = \mu\left(\frac{\mu+2}{2\mu+1}\right)^3, \quad k^2 = \mu^3\left(\frac{\mu+2}{2\mu+1}\right),$$

i.e.,
$$1-u^4v^4 \pm 2uv\,(v^2-u^2) = 0,$$

if $\lambda = v^4$, $k = u^4$. Also

$$M = \left(\frac{\mu+2}{\mu}\right)k.$$

For a quartic transformation, one form is

$$iu = \tan\,(X+Y+Z+W),$$

where
$$\tan X = i\,.\,\frac{ax+1}{x+a}, \quad \tan Y = i\,.\,\frac{akx-1}{-kx+a},$$

$$\tan Z = i\,.\,\frac{\sqrt{k}\,x+i}{i\sqrt{k}\,x+1}, \quad \tan W = i\,.\,\frac{\sqrt{k}\,x-i}{-i\sqrt{k}\,x+1}.$$

This gives
$$u = \frac{(1+\sqrt{k})^2 x\,(1+kx^2)}{1+2\sqrt{k}\,(1+\sqrt{k}+k)\,x^2+k^2x^4},$$

$$\sqrt{1-u^2} = \sqrt{1-x^2\,.\,1-k^2x^2}\,(1-kx^2) \div,$$

$$\sqrt{1-\lambda^2u^2} = 1-2\sqrt{k}\,(1-\sqrt{k}+k)\,x^2+k^2x^4 \div,$$

the denominator being the same as in u.

Also
$$\sqrt{\lambda'} = \frac{1-\sqrt{k}}{1+\sqrt{k}},$$

$$M = (1+\sqrt{k})^2,$$

$$\frac{du}{\sqrt{1-u^2\,.\,1-\lambda^2u^2}} = M\,\frac{dx}{\sqrt{1-x^2\,.\,1-k^2x^2}}.$$

Another quartic form is

$$iu = \tan\,(X+Y+Z+W),$$

where $\tan X = i \,.\, \dfrac{(\beta-1)\,x+2}{(\beta+1)\,x}$, $\tan Y = i \,.\, \dfrac{(\beta-1)\,x-2}{(\beta+1)\,x}$,

$$\tan Z = i \,.\, \frac{\alpha\beta x+1}{(2-\alpha)\,\beta x+1}, \quad \tan W = i \,.\, \frac{\alpha\beta x+1}{-(2+\alpha)\,\beta x+1},$$

i.e.,
$$u = \frac{1-(1+\alpha^2)\,\beta^2x^2+(\alpha^2+\beta^2-1)\,\beta^2x^4}{1-(3-\alpha^2)\,\beta^2x^2+(-\alpha^2+\beta^2+1)\,\beta^2x^4}.$$

Here, if constant factors be omitted,

$$(\beta+1)^2\,x^2-\{(\beta-1)\,x+2\}^2 \equiv (1-x)(1+\beta x),$$
$$(\beta+1)^2\,x^2-\{(\beta-1)\,x-2\}^2 \equiv (1+x)(1-\beta x),$$
$$\{(2-\alpha)\,\beta x+1\}^2-\{\alpha\beta x+1\}^2 \equiv x\,(1+\beta x),$$
$$\{-(2+\alpha)\,\beta x+1\}^2-\{\alpha\beta x+1\}^2 \equiv x\,(1-\beta x).$$

So that $\sqrt{1-u^2} = x\sqrt{1-x^2} \times$ rational function of x^2,

and $\sqrt{1-\lambda^2u^2} = \sqrt{1-k^2x^2} \times$ ditto.

If we put $\alpha^2 = 1+2\sqrt{k'}$, $\beta^2 = 1+k'$,

the actual form of u is

$$u = \frac{1-2\,(1+\sqrt{k'})(1+k')\,x^2+(1+\sqrt{k'})^2\,(1+k')\,x^4}{1-2\,(1-\sqrt{k'})(1+k')\,x^2+(1-\sqrt{k'})^2\,(1+k')\,x^4}.$$

This gives
$$\frac{du}{\sqrt{1-u^2\,.\,1-\lambda^2u^2}} = -\,(1+\sqrt{k'})^2\,\frac{dx}{\sqrt{1-x^2\,.\,1-k^2x^2}},$$

if
$$\sqrt{\lambda} = \frac{1-\sqrt{k}}{1+\sqrt{k'}}.$$

The above pair of quartic forms may be combined together so as to lead to multiplication by -4.

Thus $y = (1+\sqrt{k})^2\,x\,(1+kx^2) \div 1+2\sqrt{k}\,(1+\sqrt{k}+k)\,x^2+k^2x^4$

gives
$$\frac{dy}{\sqrt{1-y^2\,.\,1-\lambda^2y^2}} = (1+\sqrt{k})^2\,\frac{dx}{\sqrt{1-x^2\,.\,1-k^2x^2}},$$

where
$$\sqrt{\lambda'} = \frac{1-\sqrt{k}}{1+\sqrt{k}},$$

and $z = 1-2\,(1+\sqrt{\lambda'})(1+\lambda')\,y^2+(1+\sqrt{\lambda'})^2\,(1+\lambda')\,y^4$

$$\div\; 1-2\,(1-\sqrt{\lambda'})(1+\lambda')\,y^2+(1-\sqrt{\lambda'})^2\,(1+\lambda')\,y^4$$

gives $$\frac{dz}{\sqrt{1-z^2 \,.\, 1-k^2z^2}} = -(1+\sqrt{\lambda'})^2 \frac{dy}{\sqrt{1-y^2 \,.\, 1-\lambda^2y^2}}$$
$$= -4\frac{dx}{\sqrt{1-x^2 \,.\, 1-k^2x^2}}.$$

Part II.

Sect. 1.—Transformation of the function Θ. (Jacobi's first transformation.)

It has been shown by the foregoing results that the present theory can be used with advantage in the transformation or partial multiplication of the elliptic functions sn, cn, &c. The question arises whether it can be applied to the Θ-function. Practically, it has been already carried by me in a former note up to this point. (See *Proc. Lond. Math. Soc.*, Dec. 1883.) By a differentiation with respect to k, the equation

$$y = \sin(\theta + A + B + \ldots),$$

where $$\sin\theta = x, \quad \cos A = \frac{1-(1+a'^2)\,x^2}{1-a^2x^2}, \text{ \&c.},$$

leads to the relation

$$f(v, \lambda) = \frac{n}{M} f(u, k) + \lambda\lambda'^2 u \frac{dM}{d\lambda}$$
$$-2\frac{nkk'^2}{M^2}\frac{\text{sn}(u,k)\,\text{cn}(u,k)}{\text{dn}(v,\lambda)}\Sigma\frac{\dfrac{da'}{dk}}{1-a^2\,\text{sn}^2(u,k)},$$

if we write for shortness

$$E(v,\lambda) - \lambda'^2 v - \lambda^2\frac{\text{sn}(v,\lambda)\,\text{cn}(v,\lambda)}{\text{dn}(v,\lambda)} = f(v,\lambda),$$

v being equal to Mu.

Let $u = K$, then $v = MK = n\Lambda$, and we have

$$E(n\Lambda,\lambda) - \lambda'^2 n\Lambda = \frac{n}{M}\{E(K,k) - k'^2K\} + \lambda\lambda'^2 K\frac{dM}{d\lambda},$$

or $$E(\Lambda,\lambda) - \lambda'^2\Lambda = \frac{1}{M}\{E(K,k) - k'^2K\} + \lambda\lambda'^2\frac{K}{n}\frac{dM}{d\lambda}.$$

Introducing Jacobi's $Z(u)$ function, *i.e.*, writing

$$E(u) = Z(u) + \frac{E}{K}u,$$

the above becomes

$$Z(v,\lambda)+\frac{G}{\Lambda}v-\lambda'^2v-\lambda^2\frac{\operatorname{sn}(v,\lambda)\operatorname{cn}(v,\lambda)}{\operatorname{dn}(v,\lambda)}$$

$$=\frac{n}{M}\left\{Z(u,k)+\frac{E}{K}u-k'^2u-\dots\right\}+\lambda\lambda'^2u\frac{dM}{d\lambda}-\frac{2nkk'^2}{M^2}\frac{\operatorname{sn}u\operatorname{cn}u}{\operatorname{dn}(v,\lambda)}\Sigma\dots,$$

or
$$Z(v,\lambda)-\lambda^2\frac{\operatorname{sn}(v,\lambda)\operatorname{cn}(v,\lambda)}{\operatorname{dn}(v,\lambda)}$$

$$=\frac{n}{M}\left\{Z(u,k)-k^2\frac{\operatorname{sn}u\operatorname{cn}u}{\operatorname{dn}u}\right\}-2n\frac{kk'^2}{M^2}\frac{\operatorname{sn}u\operatorname{cn}u}{\operatorname{dn}(v,\lambda)}\Sigma\dots,$$

if
$$G=E(\Lambda,\lambda)$$

and
$$\frac{G}{\Lambda}v-\lambda'^2v=\frac{n}{M}\left(\frac{E}{K}u-k'^2u\right)+\lambda\lambda'^2u\frac{dM}{d\lambda},$$

i.e., if
$$G-\lambda'^2\Lambda=\frac{1}{M}(E-k'^2K)+\lambda\lambda'^2\frac{K}{n}\frac{dM}{d\lambda},$$

since
$$v=Mu \text{ and } n\Lambda=MK.$$

Again, introducing the Θ-function, *i.e.*, writing

$$Z(u)=\frac{\Theta'(u)}{\Theta(u)},$$

we have
$$M\frac{d}{dv}\log\Theta(v,\lambda)+M\frac{d}{dv}\log\operatorname{dn}(v,\lambda)$$

$$=n\frac{d}{du}\{\log\Theta(u)+\log\operatorname{dn}u\}-2\frac{n}{M}kk'^2\frac{\operatorname{sn}u\operatorname{cn}u}{\operatorname{dn}(v,\lambda)}\Sigma\frac{\frac{da'}{dk}}{1-a^2\operatorname{sn}^2u},$$

i.e., $\frac{d}{du}\log\{\Theta(v,\lambda)\operatorname{dn}(v,\lambda)\}=n\frac{d}{du}\log\{\Theta(u)\operatorname{dn}(u)\}-\dots$

Now, since $v=Mu$ and $\operatorname{dn}(v,\lambda)$ is a function of $\operatorname{sn}u$, it follows that the expression

$$-2\frac{n}{M}kk'^2\frac{\operatorname{sn}u\operatorname{cn}u}{\operatorname{dn}(v,\lambda)}\Sigma\frac{\frac{da'}{dk}}{1-a^2\operatorname{sn}^2u}$$

is a function of u. Denoting this, for a moment, by $\frac{\phi'(u)}{\phi(u)}$, we obtain the result $\Theta(v,\lambda)\operatorname{dn}(v,\lambda)=\{\Theta(u)\operatorname{dn}u\}^n\phi(u)$,

to a constant factor *près*, or, what is the same thing,

$$\Theta(v+\Lambda,\lambda)=\Theta^n(u+K)\phi(u).$$

If u be changed into $u+K$, and, consequently, v into $v+n\Lambda$ (*first* transformation), this gives

$$\Theta(v+\overline{n+1}\,\Lambda, \lambda) = \Theta^n(u+2K)\,\phi(u+K),$$

or, since $n+1$ is an even number,

$$\Theta(v, \lambda) = \Theta^n(u)\,\phi(u+K).$$

Hence
$$\frac{\Theta(v+\Lambda, \lambda)}{\Theta(v, \lambda)} = \frac{\Theta^n(u+K)}{\Theta^n(u)}\,\frac{\phi(u)}{\phi(u+K)},$$

i.e.,
$$\text{dn}(v, \lambda) = (\text{dn}\,u)^n\,\frac{\phi(u)}{\phi(u+K)},$$

to a constant factor *près*.

Finally, then, by comparing this with the known expression for dn (v, λ), viz.,

$$\text{dn}(v, \lambda) = \text{dn}\,u\left[1-k^2\,\text{sn}^2\left(K-\frac{2sK}{n}\right)\text{sn}^2 u\right]$$
$$\div\left[1-k^2\,\text{sn}^2\frac{2sK}{n}\,\text{sn}^2 u\right],$$

it is inferred that

$$\phi(u) = \left[1-k^2\,\text{sn}^2\frac{2sK}{n}\,\text{sn}^2(u+K)\right],$$

where [ ] denotes the product arising from the different values of the integer s extending from 1 to $\frac{1}{2}(n-1)$.

[For the value of dn (Mu, λ) in terms of dn u, &c., here used, see Prof. Cayley's "Elliptic Functions," p. 263.]

It appears, then, that we may take

$$\Theta(v+\Lambda, \lambda) = C.\Theta^n(u+K)\left[1-k^2\,\text{sn}^2\frac{2sK}{n}\,\text{sn}^2(u+K)\right],$$

where C is a constant, and consequently

$$\Theta(Mu, \lambda) = C.\Theta^n(u)\left[1-k^2\,\text{sn}^2\frac{2sK}{n}\,\text{sn}^2 u\right],$$

which is the formula given by Prof. Cayley. (See "Elliptic Functions," p. 307.)

Since $\phi(u) = [1-\alpha^2\,\text{sn}^2(u+K)]$, and consequently

$$\frac{\phi'(u)}{\phi(u)} = -\Sigma\frac{\alpha^2\dfrac{d}{du}\,\text{sn}^2(u+K)}{1-\alpha^2\,\text{sn}^2(u+K)},$$

the above method leads to the equation

$$-\frac{n}{M} k \frac{\operatorname{dn} u}{\operatorname{dn}(v, \lambda)} \Sigma \frac{\frac{da'}{dk}}{1-a^2 \operatorname{sn}^2 u} = \Sigma \frac{a^2}{\operatorname{dn}^2 u - a^2 \operatorname{cn}^2 u},$$

or, in other words, to the identity

$$-nk \Sigma \frac{\frac{da'}{dk}}{1-a^2 \operatorname{sn}^2 u} = \left(1+\Sigma \frac{2a'}{1-a^2 \operatorname{sn}^2 u}\right) \Sigma \frac{a^2}{\operatorname{dn}^2 u - a^2 \operatorname{cn}^2 u}.$$

Hence the different coefficients a, β, &c. must satisfy differential equations of the type-form

$$nk \frac{da'}{dk} + 2a^2 a' \left\{ \frac{a^2}{k'^2 - a'^4} + \Sigma \frac{\beta^2}{k'^2 - a'^2 \beta'^2} \right\} = 0.$$

Sect. 2.—Transformation of the function Θ corresponding to an imaginary root of the modular equation.

Since the other transformations, as well as the case of complete multiplication, are all included in the formula

$$f(v, \lambda) = \frac{n}{M} f(u, k) + \&c.,$$

we have an important theorem for the multiplication of Θ-functions, which, so far as I know, has not been hitherto noticed; viz.,

$$\Pi \frac{\Theta(Mu, \lambda)}{\Theta(0, \lambda)} = e^{-\mu \frac{u^2}{K^2}} \frac{\Theta nu\, \Theta^n u}{\Theta^{n+1} 0},$$

where Π denotes the product arising from the $n+1$ roots of the modular equation, and μ is a constant which will presently be determined.

If in an imaginary transformation it be assumed that

$$MK = a\Lambda + 2bi\Lambda'$$

(a an odd number), the above formula gives

$$\Theta^{n-1} 0 \,.\, \Theta(Mu, \lambda) \div \Theta^n u = \frac{\Theta(0, \lambda)}{\Theta 0} e^{-\mu \frac{u^2}{K^2}} \Pi (1 - a^2 \operatorname{sn}^2 u),$$

where $$a \equiv k \operatorname{sn} \frac{2mK + 2m'iK'}{n},$$

m and m' being certain integers, and

$$\mu = \frac{\pi b i M K}{2\Lambda} = \frac{\pi a b i}{2} - \pi b^2 \frac{\Lambda'}{\Lambda};$$

i.e., $e^{-\mu}$ is reducible to a q-function $(q = e^{-\pi\frac{\Lambda'}{\Lambda}})$.

This includes the case of a real transformation, as may be seen by putting $b = 0$, *i.e.* $\mu = 0$.

Sect. 3.—Expression for $\Theta\,(nu)$,—complete multiplication.

The equation referred to above, viz.,

$$E\,(v, \lambda) - \lambda'^2 v - \ldots = \frac{n}{M}\,\{E\,(u, k) - k'^2 u - \ldots\} + \lambda\lambda'^2 u \frac{dM}{d\lambda} - \ldots$$

(see Sect. 1, Part II.), is a most important one as regards the theory of multiplication.

Supposing $\lambda = k$ to be a possible root of the modular equation, then, since

$$M^2 = n \frac{kk'^2}{\lambda\lambda'^2} \frac{d\lambda}{dk},$$

M^2 becomes equal to n, and the relation in question reduces to

$$E\,\{\sqrt{n}.u, k\} = \sqrt{n}.E\,(u, k) + \text{an algebraical function of } \operatorname{sn} u,$$

the terms involving v and u having vanished owing to the fact that $v = \sqrt{n}.u$, and M is now independent of k.

But, having regard to the addition theorem

$$E\,(u+v) = E\,(u) + E\,(v) + \text{algebraical function of } \operatorname{sn} u \text{ and } \operatorname{sn} v,$$

it is seen that this result can only be true when n is a square number.

Given, therefore, that when n is a square number (odd) $\lambda = k$ is a root of the modular equation, it follows that

$$\Theta\,(\sqrt{n}.u, k) = C.\Theta^n\,(u)\,[1 - \alpha^2 \operatorname{sn}^2 u];$$

where

$$\alpha = k \operatorname{sn} \frac{2mK + 2m'iK}{\sqrt{n}},$$

i.e., where the different coefficients α, β, &c. include *all* the roots of $\operatorname{sn} \sqrt{n}.u = 0$, and not some only, as in the case of a transformation or partial multiplication.

If we write n^2 for n, we have

$$\Theta\,(nu, k) = C.\Theta^{n^2}\,(u)\,[1 - \alpha^2 \operatorname{sn}^2 u];$$

$$\alpha = k \operatorname{sn} \frac{2mK + 2m'iK'}{n},$$

and C = a constant.

The complete multiplication by n, therefore, includes all the coefficients which occur in the different transformations or partial multiplications of order n, and hence there must be the theorem

$$\Pi \frac{\Theta\,(Mu,\,\lambda)}{\Theta\,(0,\,\lambda)} = e^{-\Sigma\mu\frac{u^2}{K^2}} \frac{\Theta nu\,\Theta^n u}{\Theta^{n+1} 0},$$

where Π denotes the product arising from the $n+1$ different roots of the modular equation, and

$$\Sigma\mu = \Sigma \frac{\pi biMK}{2\Lambda}.$$

Sect. 4.—Transformation of the function Θ for the irrational equation $y = \sin\,(L+A+B+C+\dots)$. (See General Outline.)

Here the question suggests itself, what are the corresponding results for $\Theta\,(Mu,\,\lambda)$ and $\Theta\,(nu,\,k)$ in the case of the even irrational transformation or analogue of $y = \sin 2nu$? As regards the former function, I have arrived, from the expression

$$y = \sin\,(L+\Sigma A),$$

in a similar manner to the above, at the formulæ

$$\Theta\,(v+\Lambda,\,\lambda) = C_1.\Theta^{2n}\,(u+K)\,.\,\Pi\left\{1-k^2\,\mathrm{sn}^2\,\frac{2t-1}{2n}\,K\,\mathrm{sn}^2\,(u+K)\right\},$$

$$\Theta\,(v,\,\lambda) = C_2.\Theta^{2n}\,(u)\,\mathrm{dn}\,u\,.\,\Pi\left\{1-k^2\,\mathrm{sn}^2\,\frac{sK}{n}\,\mathrm{sn}^2\,u\right\},$$

where $v = Mu$, C_1, C_2 constants, s an integer from 1 to $n-1$, and t an integer from 1 to n; also

$$\mathrm{dn}\,(v,\,\lambda) = \mathrm{dn}\,u\,.\,\Pi\left\{\frac{1-k^2\,\mathrm{sn}^2\,\frac{2t-1}{2n}\,K\,\mathrm{sn}^2\,u}{1-k\ \mathrm{sn}^2\,\frac{tK}{n}\,\mathrm{sn}^2\,u}\right\}.$$

Since there is here no root $\lambda = k$ of the modular equation, the case of complete multiplication is different. This can be accounted for by the fact that an irrational transformation cannot of itself lead to complete multiplication. The most convenient formula that occurs to me is one derived from the secondary transformation noticed in Section 2, Part I. For example,

$$\mathrm{am}\,(2u,\,k) = X_1+X_2,$$

where $$\sin X_1 = \frac{(1+k)\,\mathrm{sn}\,u}{1+k\,\mathrm{sn}^2\,u},\quad \sin X_2 = \frac{(1-k)\,\mathrm{sn}\,u}{1-k\,\mathrm{sn}^2\,u}.$$

Hence, by a differentiation with respect to k, we have, after some reductions,
$$\Theta(2u) = C.\Theta^4(u)(1-k^4\operatorname{sn}^4 u).$$
For $2n = 4$, the formula is
$$\operatorname{am}(4u, k) = X_1+X_2+X_3+X_4,$$
where
$$\sin X_1 = (1+\sqrt{k})^2 \frac{\operatorname{sn} u\,(1+k\operatorname{sn}^2 u)}{1+2\sqrt{k}\,(1+\sqrt{k}+k)\operatorname{sn}^2 u+k^2\operatorname{sn}^4 u},$$
$$\sin X_2 = (1+\sqrt{k})^2 \frac{\operatorname{sn} u\,(1+k\operatorname{sn}^2 u)}{1-2\sqrt{k}\,(1-\sqrt{k}+k)\operatorname{sn}^2 u+k^2\operatorname{sn}^4 u},$$
$$\sin X_3 = (1-\sqrt{k})^2 \frac{\operatorname{sn} u\,(1+k\operatorname{sn}^2 u)}{1+2\sqrt{k}\,(1+\sqrt{k}+k)\operatorname{sn}^2 u+k^2\operatorname{sn}^4 u},$$
$$\sin X_4 = (1-\sqrt{k})^2 \frac{\operatorname{sn} u\,(1+k\operatorname{sn}^2 u)}{1-2\sqrt{k}\,(1-\sqrt{k}+k)\operatorname{sn}^2 u+k^2\operatorname{sn}^4 u}.$$

But, with the exception of the case of the prime number 2, these functions are of only secondary interest, and I do not consider the subject further than to remark that there seems to be the formula
$$\Theta(2nu) = C.\Theta^{4n^2}(u)\,\Pi\,(1-a^2\operatorname{sn}^2 u),$$
where
$$a \equiv k\operatorname{sn}\frac{2mK+(2m'+1)\,iK'}{2n}, \quad C = 1 \div \Theta^{4n^2-1}(0).$$

Appendix.

On a Problem in Jacobi's "Fundamenta Nova," p. 75.

It appears to me that the principle of duality referred to in Lemma 5, Sect. 2, Part I., of the present Note, throws light on a problem which seems to have engaged Jacobi's attention, but to have been left unsolved by him. (See *Fundamenta Nova*, p. 75.)

This is the problem of determining the integers which present themselves in the relation between Λ and K corresponding to any imaginary transformation of an odd order, viz.,
$$y = \frac{x\,(1, x^2)^{\frac{1}{2}(n-1)}}{(1, x^2)^{\frac{1}{2}(n-1)}}.$$

When n is an odd prime, the equation which gives λ in terms of k ($k<1$) is known to be of the $n+1^{\text{th}}$ degree, having only two real values of $\lambda<1$. To these correspond real transformations, and to the remaining roots $n-1$ imaginary transformations.

For the first transformation, or the real root λ, the modular equation is $n\frac{\Lambda}{\Lambda'}=\frac{K}{K'}$, and for the second real root λ_1, it is $n\frac{K}{K'}=\frac{\Lambda_1}{\Lambda_1'}$. These are derived from the equations $n\Lambda = M.K$, $\Lambda' = M.K'$, and $n\Lambda_1' = M_1.K'$, $\Lambda_1 = M_1.K$, which show that the first and second transformations in question may be combined together so as to give complete multiplication by n; *i.e.*, that λ, k, M can be changed into k, λ_1, and $\frac{n}{M_1}$, respectively, or λ_1, k, M into k, λ, $\frac{n}{M}$.

But for a pair of imaginary transformations this is not always the case. For example, when $k=0$ or 1, there can be no multiplication, since the pair then reduce to $y=x$.

Excluding, then, the cases of non-multiplication, and taking λ_2 and λ_3 to be a conjugate pair of imaginary roots, we may assume relations of the form,

$$\left.\begin{aligned} M_2K &= a\Lambda_2+ib\Lambda_2' \\ M_2K' &= a'\Lambda_2'+ib'\Lambda_2 \end{aligned}\right\},\qquad \left.\begin{aligned} M_3K &= a'\Lambda_3+ib'\Lambda_3' \\ M_3K' &= a\Lambda_3'+ib\Lambda_3 \end{aligned}\right\},$$

or $$\left.\begin{aligned} (aa'+bb')\,\Lambda_2 &= M_2(a'K-ibK') \\ (aa'+bb')\,\Lambda_2' &= M_2(aK'-ib'K) \end{aligned}\right\},\quad \left.\begin{aligned} (aa'+bb')\,\Lambda_3 &= M_3(aK-ib'K') \\ (aa'+bb')\,\Lambda_3' &= M_3(a'K'-ibK) \end{aligned}\right\},$$

where a, a' are odd, and b, b' even numbers.

That there are relations of the above form may be inferred from the equation $$y=\frac{x\,(1,\ x^2)^{\frac{1}{2}(n-1)}}{(1,\ x^2)^{\frac{1}{2}(n-1)}}.$$

Here $$\sqrt{1-y^2}=\sqrt{1-x^2}\times\text{rational function of } x,$$

so that, if $x=1$, then $y=\pm 1$, and

$$\int_0^{\pm 1}\frac{dy}{\sqrt{1-y^2\,.\,1-\lambda_2^2y^2}}=M_2\int_0^1\frac{dx}{\sqrt{1-x^2\,.\,1-k^2x^2}},$$

or $$\operatorname{sn}(a\Lambda_2+ib\Lambda_2',\ \lambda_2)=\pm 1,$$

if a be an odd, and b an even number.

Again, changing λ_2, k, M_2 into k, λ_3, M_3 respectively, we have

$$\frac{n}{M_3}\Lambda_3=aK+ibK',\quad \frac{n}{M_3}\Lambda_3'=a'K'+ib'K.$$

These values of Λ_3, Λ_3' are compatible with

$(aa'+bb')\,\Lambda_3 = M_3\,(aK+ibK')$ and $(aa'+bb')\,\Lambda_3' = M_3\,(a'K'-ibK)$,

provided that $\qquad aa'+bb' = n, \quad b+b' = 0.$

Conversely, when these relations between the numbers a, a, b, b' are satisfied, the pair of imaginary transformations may be so combined as to give the complete multiplication by n.

It thus appears that

$$\left.\begin{aligned} M_2.K &= \frac{1}{2}(n\pm 1)\,\Lambda_2 + \frac{i}{2}(n\mp 1)\,\Lambda_2' \\ M_2.K' &= -\frac{i}{2}(n\mp 1)\,\Lambda_2 + \frac{1}{2}(n\pm 1)\,\Lambda_2' \end{aligned}\right\},$$

$$\left.\begin{aligned} M_3.K &= -\frac{i}{2}(n\mp 1)\,\Lambda_3' + \frac{1}{2}(n\pm 1)\,\Lambda_3 \\ M_3.K' &= \frac{1}{2}(n\pm 1)\,\Lambda_3' + \frac{i}{2}(n\mp 1)\,\Lambda_3 \end{aligned}\right\},$$

where the upper and lower signs are to be taken according as $\frac{1}{2}(n+1)$ or $\frac{1}{2}(n-1)$ is odd, is a possible solution.

For a pair of imaginary roots (and complete multiplication) the modular equations, then, may be taken to be

$$\frac{\Lambda_2}{\Lambda_2'} = \frac{-ibK'+a'K}{aK'+ibK}, \qquad \frac{\Lambda_3}{\Lambda_3'} = \frac{aK+ibK'}{-ibK+a'K'},$$

where $\qquad aa'-b^2 = \pm n.$

In particular, if $K = K'$, or $k = k' = \dfrac{1}{\sqrt{2}}$, then

$$\Lambda_2 = \Lambda_3', \;\; \Lambda_2' = \Lambda_3, \quad i.e., \;\; \lambda_2^2+\lambda_3^2 = 1.$$

There seems to be also a special form when $n \equiv a^2+b^2$, *i.e.*, when $n = 5$, 13, 17, &c.

In this case we may have

$$\left.\begin{aligned} M_2.K &= a\Lambda_2 + ib\Lambda_2' \\ M_2.K' &= ib\Lambda_2 + a\Lambda_2' \end{aligned}\right\}, \qquad \left.\begin{aligned} M_3.K &= -ib\Lambda_3' + a\Lambda_3 \\ M_3.K' &= a\Lambda_3' - ib\Lambda_3 \end{aligned}\right\},$$

(a an odd and b an even number), and when $K' = K$, or $k = \dfrac{1}{\sqrt{2}}$, and also $M_2 = a+ib$, $M_3 = a-ib$, we have $\Lambda_2 = \Lambda_2'$, $\Lambda_3 = \Lambda_3'$, or $\lambda_2 = \lambda_3 = \dfrac{1}{\sqrt{2}}$ for a particular pair of roots.

It should be mentioned also that, since all the results of the present theory point to an addition theorem of the form

$\pm \text{am}\,(Mu, \lambda) + \text{am}\,(M_1 u, \lambda_1) + \Sigma\, \text{am}\,(M_2 u, \lambda_2) = \text{am}\,(nu, k) \pm n\, \text{am}\,(u, k)$,

there must be another condition to be satisfied by the numbers a, b, c. In fact, we must have

$$\Sigma\, \text{am}\,(M_2 K, \lambda_2) = (n-1)\frac{\pi}{2},^*$$

where the summation applies to the $\frac{1}{2}(n-1)$ pairs of imaginary roots λ_2, λ_3, &c.

The following are possible forms for $n = 3$ and 5. I have also considered the case of $n = 7$, but, as there are here six imaginary roots, the results are somewhat long to write down. I content myself with giving a pair.

Modular equations corresponding to the imaginary transformations

$$y = \frac{x\,(1, x^2)^{\frac{1}{2}(n-1)}}{(1, x^2)^{\frac{1}{2}(n-1)}},$$

when $n = 3$, 5, and 7.

Case of $n = 3$.

$$\left.\begin{aligned} M_2.K &= \Lambda_2 + 2i\Lambda_2' \\ M_2.K' &= -.2i\Lambda_2 + \Lambda_2' \end{aligned}\right\}, \qquad \left.\begin{aligned} M_3.K &= -2i\Lambda_3' + \Lambda_3 \\ M_3.K' &= \Lambda_3' + 2i\Lambda_3 \end{aligned}\right\}.$$

Change λ_2 into k, k into λ_3, and M_3 into $-\frac{3}{M_3}$, then

$$\left.\begin{aligned} \Lambda_3 &= -\frac{M_3}{3}(K + 2iK') \\ \Lambda_3' &= -\frac{M_3}{3}(-2iK + K') \end{aligned}\right\}.$$

Change λ_3 into k, k into λ_2, and M_3 into $-\frac{3}{M_2}$, then

$$\left.\begin{aligned} \Lambda_2 &= -\frac{M_2}{3}(-2iK' + K) \\ \Lambda_2' &= -\frac{M_2}{3}(K' + 2iK) \end{aligned}\right\}.$$

We have also $\text{am}\,(\Lambda_2 + 2i\Lambda_2', \lambda_2) + \text{am}\,(\Lambda_3 - 2i\Lambda_3', \lambda_3) = 2\frac{\pi}{2}$.

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* In the proof of the above addition theorem for $n = 3$, it was assumed in a former paper that $M_2.K = \Lambda_2$ and $M_3.K = \Lambda_3$. This assumption is too special, and, indeed, is not necessary for the argument. All that is necessary is that

$$\text{am}\,(M_2.K, \lambda_2) + \text{am}\,(M_3.K, \lambda_3) = 2\frac{\pi}{2},$$

in the case ($n = 3$) referred to.

Case of $n = 5$.

$$(\alpha)\quad \left.\begin{aligned} M_2 . K &= -(\Lambda_2 + 2i\Lambda_2') \\ M_2 . K' &= -(2i\Lambda_2 + \Lambda_2') \end{aligned}\right\},\qquad \left.\begin{aligned} M_3 . K &= 2i\Lambda_3' - \Lambda_3 \\ M_3 . K' &= -\Lambda_3' + 2i\Lambda_3 \end{aligned}\right\},$$

$$(\beta)\quad \left.\begin{aligned} M_4 . K &= 3\Lambda_4 + 2i\Lambda_4' \\ M_4 . K' &= -2i\Lambda_4 + 3\Lambda_4' \end{aligned}\right\},\qquad \left.\begin{aligned} M_5 . K &= -2i\Lambda_5' + 3\Lambda_5 \\ M_5 . K' &= 3\Lambda_5' + 2i\Lambda_5 \end{aligned}\right\},$$

$$\begin{aligned} \text{am}\,(-\Lambda_2 - 2i\Lambda_2', \lambda_2) + \text{am}\,(2i\Lambda_3' - \Lambda_3, \lambda_3) + \text{am}\,(3\Lambda_4 + 2i\Lambda_4', \lambda_4) \\ + \text{am}\,(3\Lambda_5 - 2i\Lambda_5') = 4\,\frac{\pi}{2}. \end{aligned}$$

Change λ_2 into k, k into λ_3, and M_2 into $\frac{5}{M_3}$, then

$$\left.\begin{aligned} -\Lambda_3 &= \frac{M_3}{5}(K + 2iK') \\ -\Lambda_3' &= \frac{M_3}{5}(2iK + K') \end{aligned}\right\},$$

and so on for the other roots.

Case of $n = 7$.

$$\left.\begin{aligned} M_2 . K &= -3\Lambda_2 + 4i\Lambda_2' \\ M_2 . K' &= -4i\Lambda_2 - 3\Lambda_2' \end{aligned}\right\},\qquad \left.\begin{aligned} M_3 . K &= -4i\Lambda_3' - 3\Lambda_3 \\ M_3 . K' &= -3\Lambda_3' + 4i\Lambda_3 \end{aligned}\right\}$$

Change λ_2 into k, k into λ_3, and M_2 into $-\frac{7}{M_3}$, then

$$\left.\begin{aligned} -\frac{7}{M_3}\Lambda_3 &= -3K + 4iK' \\ -\frac{7}{M_3}\Lambda_3' &= -4iK - 3K' \end{aligned}\right\},$$

and so on.

On Sphero-Cyclides. By Henry M. Jeffery, F.R.S.

[*Read Nov.* 13*th*, 1884.]

1. These spherical quartics are the lines of intersection of spheres both with cyclides and quadrics. M. Laguerre, who first pointed out their genesis (*infra*, §4), designated them Anallagmatic Spherical Curves, because they are unaltered, when inverted from any of its four poles of inversion. (Chasles, "Rapport sur les progrès de la Géometrie," p. 315.) Under the name of sphero-quartics, their properties have been studied by Dr. Casey ("Cyclides and Sphero-Quartics," *Phil. Trans.*, 1871, pp. 585—721). But, since they are only a species of spherical binodal quartics, and do not include all the intersections of quartic surfaces with spheres, their name is here altered. They might be also called spherical quadro-quadrics or sphero-quadrics (§6).

2. Sphero-cyclides, being binodal, are curves of the eighth class, and have two double, and four single, foci: the former are the two single foci of the dirigent or focal sphero-conics, from which they are generated. If the two double foci coincide in a quadruple focus, the cyclide is known as a Sphero-Cartesian (Casey, p. 677). These curves may have an additional node or cusp, whereby the class is reduced to the sixth or fifth respectively.

3. Sphero-cyclides have two double cyclic arcs, which are the single cyclic arcs of the complementary or polar conics, of which cyclic arcs the single foci of the focal conics or the double foci of the sphero-cyclides are the spherical centres or quadrantal poles. Sphero-Cartesians have each a quadruple cyclic arc, whose spherical centre is the quadruple focus, or centre of the dirigent circle, from which it is generated. This conjugate property of double arcs and double cyclic arcs is common to all spherical curves (*Quarterly Math. Journal*, Vol. xv., p. 140).

4. A sphero-cyclide may be generated in four different ways, as the envelope of a variable small circle, whose centre moves on a dirigent or focal sphero-conic (F), and which cuts a fixed small circle (J) orthogonally. (Laguerre, *Bulletin de la Société Philomathique*, 1867; Casey, §41, Cor.)

These dirigent conics (F) are doubly confocal; and the fixed circles (J) are mutually orthotomic, and all the eight figures are interdependent. The centres of the four (J) circles are the vertices of the

quadrangle, in which any J and F pair intersect, and, taken three and three together, are the angular points of triangles, which are self-conjugate with respect both to the (F) conics and (J) circles. Each triad of centres has the fourth for the orthocentre of the triangle constituted by them; and each of these four triangles is self-conjugate in respect to one of the four circles and its corresponding focal conic.

The confocal conics are thus also interdependent. The twelve points in which the sides of the quadrilateral circumscribed about any pair J and F intersect, lie by tetrads on the three remaining (F) focal conics.

The line of nodes in the sphero-cyclide is the polar of the centre of any (J) circle with respect to the corresponding (F) focal conic.

5. The three anallagmatic congeners, the cyclide, the sphero-cyclide, and the bicircular quartic, constitute a geometrical trilogy, as exhibited by Professor Casey in his two classical memoirs.

Dr. Hart has shown analytically how the bicircular is generated from each of the four (F) conics (*Proceedings*, Vol. XI., pp. 143—151), and has promised this Society the corresponding memoir on the Five Focal Quadrics of a Cyclide (Vol. XII., p. 109), the MS. of which he has allowed me to see and copy. It is hoped he will shortly publish it.

Following his steps, I have investigated by spherical coordinates the generation of the sphero-cyclide from each of its four focal sphero-conics, and thereby hope to complete the series of the trilogy.

The singular forms of the curve will be considered, and a method given for finding its points of undulation, and therefrom its points of inflexion generally.

6. The equation to the sphero-cyclide is derived from those to the cyclide and quadric, by transformation of coordinates.

Let OAB be an octant of a sphere, whose centre is any origin of coordinates for the cyclide, and whose radius is unity.

Take O for the origin of spherical coordinates in Gudermann's system.

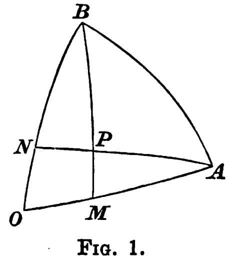

FIG. 1.

$BP = \theta,\ AM = \phi:\ OM = X,\ ON = Y,\ OP = R.$

Cartesian coordinates are thus transformed to spherical:

$$x = \sin\theta\cos\phi = \tan X\cos R,$$
$$y = \sin\theta\sin\phi = \cos R,$$
$$z = \cos\theta = \tan Y\cos R.$$

($\tan X$, $\tan Y$ are usually written X, Y, for brevity).

The equation to the cyclide in Cartesian coordinates is

$$a\,(x^2+y^2+z^2)^2+b\,(x^2+y^2+z^2)(u_1+u_0)+v_2+v_1+v_0=0,$$

where v_2; v_1, u_1; v_0, u_0 denote quadric and linear functions of x, y, z, and constants.

Let the same symbols v_2', ... denote the same functions of $\tan X$, 1, $\tan Y$.

$$a\sec^2 R+b\sec R\,(u_1'+u_0'\sec R)+v_2'+v_1'\sec R+v_0'\sec^2 R=0.$$

After dropping the accents, we obtain the transformed equation to the sphero-cyclide

$$\{(a+bu_0+v_0)\sec^2 R+v_2\}^2=\sec^2 R\,(bu_1+v_1)^2.$$

The curve is binodal, and is touched by the imaginary great circle ($\sec^2 R=1+\tan^2 X+\tan^2 Y=0$) in four points $\sec^2 R=0$, with $v_2=0$; it has for two nodes the two points $bu_1+v_1=0$, with $(a+bu_0+v_0)\sec^2 R+v_2=0$.

The equation to the quadric is derived from that to the cyclide, when $a=0$, $b=0$; for the corresponding sphero-cyclide,

$$\{v_0(1+X^2+Y^2)+v_2\}^2=(1+X^2+Y^2)\,v_1^2,$$

If $v_1=0$, or the quadric is referred to its centre as origin, the sphero-cyclide becomes two coincident sphero-conics.

Cor.—If $v_2\equiv c\,(x^2+z^2)+dy^2$, or the cyclide has the imaginary circle at infinity as a cuspidal edge, it is called by Dr. Casey a Sphero-Cartesian. Its equation in spherical coordinates becomes

$$\{(v_0+c)(1+X^2+Y^2)+d-c\}^2=v_1^2\,(1+X^2+Y^2).$$

It is thus recognised to be the intersection of a sphere and a quadric of revolution (Casey, §235).

The following theorems are preliminary; and it is necessary to premise that, in two-point coordinates, x, y denote $\tan x$, $\tan y$, and in the three-point system α, β, γ; p, q, r denote the sines of those arcs; a, b, c also represent the sines of the arcs of the triangle of reference:—

7. Two spherical small circles are mutually orthotomic, if $\cos\rho_1\cos\rho_2=\cos D$; where the symbols denote their spherical radii and the mutual distance of their centres. For in that case the centres and either point of intersection constitute a right angle.

8. To find the condition that two small circles intersect orthogonally. First, let their equations in three-point coordinates be

$$d\alpha+e\beta+f\gamma=g,\quad l\alpha+m\beta+n\gamma=h.$$

Then $\cos\rho_1\sqrt{\Sigma}(d^2-2ef\cos A) = g$; $\cos\rho_2\sqrt{\Sigma}(l^2-2mn\cos A) = h$;

$$\cos D\sqrt{\Sigma}(d^2-2ef\cos A)\,.\,\sqrt{\Sigma}(l^2-2mn\cos A)$$
$$= d(l-m\cos C-n\cos B)+\ldots$$

The required condition is

$$gh = dl+em+fn-(en+fm)\cos A-(fl+dn)\cos B-(dm+el)\cos C.$$

Next, let their equations in two-point (Gudermann's) coordinates be

$$1+ax+by = g\sqrt{(1+x^2+y^2)},\quad 1+cx+dy = h\sqrt{(1+x^2+y^2)}.$$

For mutual orthotomy, it is necessary that

$$1+ac+bd = gh.$$

9. If ABC be a spherical triangle, and O its orthocentre, then the four small circles which have A, B, C, O for their centres are mutually orthotomic, if

$$\cos a\cos\delta_1 = \cos b\cos\delta_2 = \cos c\cos\delta_3 = \sqrt{(\cos a\cos b\cos c)},$$
$$\cos\delta_4\sqrt{\Sigma}(\tan^2 A+2\tan B\tan C\cos a)$$
$$= \tan A\tan B\tan C\sqrt{(\cos a\cos b\cos c)}.$$

The radii are denoted by δ_1, δ_2, δ_3, δ_4.

The propositions in the first line are evident from § 7.

In like manner,

$$\cos a\cos\delta_4 = \cos BO\cos\delta_3 = \cos CO\cos\delta_2 = \sqrt{(\cos a\cos BO\cos CO)}.$$

The proposition in the second line is established by knowing the distance (δ) between two points from the formula

$$\sin^2 b\sin^2 c\sin^2 A\cos\delta = \Sigma(\alpha\alpha_1\sin^2 a)+\Sigma[\sin b\sin c\cos a(\beta\gamma_1+\beta_1\gamma)].$$

At the orthocentre

$$\alpha\cos A = \beta\cos B = \gamma\cos C.$$

Hence $\quad\cos AO\sqrt{\Sigma}(\tan^2 A+2\tan B\tan C\cos a)$

$$= \tan A+\tan B\cos c+\tan C\cos b = \tan A\tan B\tan C\cos b\cos c.$$

By symmetry,

$$\cos a\cos AO = \cos b\cos BO = \cos c\cos CO.$$

Hence $\cos\delta_4 = \sec\delta_1\cos AO = \sec\delta_2\cos BO = \sec\delta_3\cos CO$,

and the circles are mutually orthotomic. Since

$$\Sigma(\tan^2 A+2\tan B\tan C\cos a) = \mu^2\tan^2 A\tan^2 B\tan^2 C,$$

where $\mu^2 = 2\cos a \cos b \cos c - 1 + (6V)^2 \operatorname{cosec}^2 A \operatorname{cosec}^2 B \operatorname{cosec}^2 C$,

$$\mu \cos AO = \cos b \cos c,$$

and $$\mu \cos \delta_4 = \sqrt{(\cos a \cos b \cos c)}.$$

The analogue to this theorem for Plane Geometry is given by Dr. Casey (*Sequel to Euclid*, p. 108).

Note.—By $6V$ will be hereinafter denoted six times the volume of a certain tetrahedron constituted by three radii of the sphere, and the connectors of their extremities, so that the fundamental relation is

$$6V = bc \sin A = \sqrt{\Sigma (a^2\alpha^2 + 2bc\,\beta\gamma \cos a)}.$$

10. To find the discriminant of the binary quartic

$$(fx^2 + 2gxy + hy^2)^2 = (ux^2 + 2w'xy + vy^2)(sx + ty)^2.$$

Let A, B be invariants of single quadrics

$$A = fh - g^2, \qquad B = uv - w'^2.$$

C, D, E are invariants of systems of two quadrics

$$C = s^2v - 2stw' + t^2u, \qquad D = s^2h - 2stg + t^2f,$$

$$E = uh - 2w'g + vf,$$

also $F = u\,(sh - tg)^2 - 2w'\,(sh - tg)(sg - tf) + v\,(sg - tf)^2 = DE - AC$.

The function F occurs in investigating I_4 by symbolical methods,

$$\left(u\frac{d^2}{dy^2} - 2w'\frac{d^2}{dy\,dx} + v\frac{d^2}{dx^2}\right)\left(s\frac{d}{dy} - t\frac{d}{dx}\right)^2 (fx^2 + 2gxy + hy^2)$$

$$= \underline{|4}\,(F + \tfrac{1}{3}AC).$$

If I_4, I_6 denote the quartic and sextic invariants of the given quartic, they can be expressed in terms of the subordinate invariants:

$$3I_4 = 4\,(A + \tfrac{1}{4}C)^2 - 3DE,$$

$$27I_6 = 8\,(A + \tfrac{1}{4}C)^3 - 9DE\,(A + \tfrac{1}{4}C) + \tfrac{27}{4}BD^2,$$

$$(I_4)^3 - 27\,(I_6)^2 = E^2\left[(A + \tfrac{1}{4}C)^2 - DE\right] - 4B\,(A + \tfrac{1}{4}C)^3 + \tfrac{9}{2}BDE\,(A + \tfrac{1}{4}C)$$
$$- \tfrac{27}{16}B^2D^2.$$

This factorial form will be employed to prove that sphero-cyclides have two double and four single foci.

11. To transform from a sphero-conic to a sphero-cyclide.

I. If O be any origin of coordinates, p the perpendicular arc drawn from it on any tangent arc of the conic, its equation in two-line coordinates is

$$(u \cos^2\theta + 2w' \cos\theta \sin\theta + v \sin^2\theta) \cot^2 p$$
$$+ 2(u' \sin\theta + v' \cos\theta) \cot p + w = 0.$$

Let r denote a corresponding arc of the sphero-cyclide, and δ a constant. The formulæ of quadric transformation may take either of the forms (Casey, § 24),

$$\cos p = \cos(p-r)\cos\delta, \quad \cot p = \frac{\tan r}{\sec r \sec\delta - 1} \quad \text{.........(1)}.$$

The transformed equation denotes the sphero-cyclide in two-point coordinates,

$$ux^2 + 2w'xy + vy^2 + 2(u'y + v'x)(\sec r \sec\delta - 1) + w(\sec r \sec\delta - 1)^2,$$
$$(\sec^2 r = 1 + \tan^2 r = 1 + x^2 + y^2).$$

Formulæ of inversion are derived from (1),

$$\sec\delta = \cos r + \sin r \tan p.$$

Denote by r_1, r_2 the vector arcs of two conjugate points P_1, P_2,

$$\cos r_1 \cos r_2 = \sec^2\delta \cos^2 p - \sin^2 p, \quad \cos r_1 + \cos r_2 = 2 \sec\delta \cos^2 p\,;$$

$$\tan\frac{r_1}{2}\tan\frac{r_2}{2} = \frac{\sec\delta - 1}{\sec\delta + 1} = \tan^2\frac{\delta}{2} \quad \text{..................(2)};$$

O is therefore a centre of inversion, such that the curve and its equation are unaltered, when $\cot\frac{R}{2}\tan^2\frac{\delta}{2}$ is substituted for $\tan\frac{r}{2}$.

The centre of inversion O and the radius δ are arbitrary; but, when they are once fixed, the other centres A, B, C, and the other constants δ_1, δ_2, δ_3 are mutually related by the coorthotomic conditions of § 9.

The two conjugate points P_1, P_2 are the points of intersection in two consecutive positions of the generating circle, which cuts orthogonally the four fixed (J) circles, whose centres are A, B, C, O.

II. Dr. Casey has also assigned a remarkably elegant mode of transformation for three-point coordinates. (Casey, § 40.)

If U, V, W denote in three-point coordinates the fixed coorthotomic circles J_1, J_2, J_3, and if the dirigent focal sphero-conic (F) be defined by the tangential equation

$$(a, b, c, f, g, h \between p, q, r)^2 = 0,$$

then the sphero-cyclide, thence generated, has the identical form of equation $(a, b, c, f, g, h \between U, V, W)^2 = 0.$

The triangle of reference is constituted by the centres of the (J) circles. It should be premised that, if the coordinates denote a point not on a circle U,

$$U = \cos AP - \cos \delta_1.$$

This follows from the equation to a circle, whose centre is (l, m),

$$U \equiv \frac{lx + my + 1}{\sqrt{(l^2+m^2+1)}\ \sqrt{(x^2+y^2+1)}} - \cos \delta_1 = 0.$$

It denotes the distance of that point from the plane of the small circle.

To prove that $p : q : r :: U : V : W$. Let F be a dirigent focal conic, O the corresponding centre of inversion; so that, by § 11 (I.),

$$\cos OT = \cos \delta \cos PT = \cos \delta \cos P'T.$$

Let this be written

$$\cos P = \cos \delta \cos (P - R) \ \ldots\ldots \ (1).$$

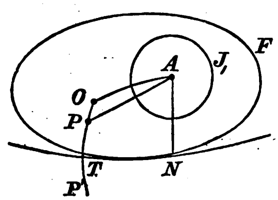

Fig. 2.

Let A be the centre, and δ_1 the radius of (J_1), one of the other three centres of inversion; $AN = p$: TO, AN, when produced, form an angle θ.

$$\cos AO = \sin P \sin p + \cos P \cos p \cos \theta,$$

$$\cos AP = \sin p \sin (P - R) + \cos p \cos (P - R) \cos \theta.$$

Eliminate θ by the aid of (1),

$$\cos AO \sec P - \cos AP \sec (P - R) = \sin p \sin R \sec P \sec (P - R).$$

Hence $U = \cos AP - \cos \delta_1 = \cos AP - \cos AO \sec \delta$, from orthotomy (§ 9),

$$= \cos AP - \cos AO \cos (P - R) \sec P$$

$$= -\sin p \sin R \sec P,$$

V, W are like multiples of $\sin q$, $\sin r$; so that, after dropping the word sin, as stated in § 6, the formulæ for quadric transformation are

$$p : q : r = U : V : W.$$

I. *On the General Form of Sphero-Cyclides.*

12. To determine the equations to the four fixed co-orthoto

small circles J, J_1, J_2, J_3, and the corresponding doubly confocal dirigent and focal sphero-conics (F), by means of which the sphero-cyclide is generated in four different ways.

Let O, A, B, C, the orthocentres of the spherical triangles ABC, BOC, COA, AOB, be the centres of four fixed coorthotomic circles J, J_1, J_2, J_3. The same centres, taken by triads, are the angular points of triangles, which are self-conjugate with respect to the four (J) circles, and to the four corresponding (F) dirigent sphero-conics.

Let ABC be first taken as the triangle of reference; then the equation in spherics to the circle (J), with respect to which it is self-conjugate, is

$$(J) \quad \alpha^2 \cos A \tan a + \beta^2 \cos B \tan b + \gamma^2 \cos C \tan c = 0,$$

or, by the aid of the fundamental relation

$$(6V)^2 = \Sigma\,(a^2\alpha^2 + 2bc\,\beta\gamma \cos a),$$

$$(J) \quad \alpha \tan a + \beta \tan b + \gamma \tan c = 6V\sqrt{(\sec a \sec b \sec c)}.$$

In like manner (J_1), one of the other three circles, may be denoted in four different forms of the same equation:

$$(J_1) \quad \alpha^2 \cos A \tan a + \beta^2 \cos B \tan b + \gamma^2 \cos C \tan c$$
$$-2\alpha \cos A\,(\alpha \tan a + \beta \tan b + \gamma \tan c) + a^2 \cos^2 A \tan a \tan b \tan c = 0,$$

$$(J_1) \quad \alpha \tan a + \beta \tan b + \gamma \tan c - 6V\sqrt{(\sec a \sec b \sec c)}$$
$$+ a \tan a \tan b \tan c \cos A = 0,$$

$$(J_1) \quad a\alpha + b\beta \cos c + c\gamma \cos b = 6V \sec a \sqrt{(\cos a \cos b \cos c)},$$

$$(J_1) \qquad \alpha' = \sec a \sqrt{(\cos a \cos b \cos c)},$$

if α', β', γ' denote the coordinates of a point in (J_1), with respect to the polar triangle of ABC.

This last form determines independently the radius of the (J_1) circle given in § 9,

$$\cos a \cos \delta_1 = \sqrt{(\cos a \cos b \cos c)}.$$

In like forms the equations to the two remaining circles J_2, J_3 may be written. From the forms of all four equations to these circles, it is recognised that, of their twenty-four points of intersection, twelve lie on the perpendiculars OA, OB, OC, BC, CA, AB. These arcs are therefore their radical axes; and O, A, B, C are the radical centres of the four triads of circles.

The equation to some one dirigent conic (F), that corresponding to

the (J) circle, is assumed to be

$$la^2\alpha^2+mb^2\beta^2+nc^2\gamma^2=0,$$

or, in three-line coordinates,

$$\frac{p^2}{l}+\frac{q^2}{m}+\frac{r^2}{n}=0.$$

The equations to the confocal conics will be expressed in terms of (F) in § 15.

13. To find the equation to the sphero-cyclide, by considering it as the envelope of a circle, whose centre moves on the focal conic (F), and which cuts the circle (J) orthogonally.

Let (λ, μ, ν) be the centre of the variable circle; its equation is (§ 9)

$$(6V)^2\cos r = a\lambda\,(a\alpha+b\beta\cos c+c\gamma\cos b)+\ldots,$$

or, if it be referred to the polar triangle of ABC,

$$6V\cos r = a\lambda\alpha'+b\mu\beta'+c\nu\gamma'.$$

For the fixed (J) circle (§ 12),

$$\alpha\tan a+\beta\tan b+\gamma\tan c = 6V\sqrt{(\sec a\sec b\sec c)}.$$

From the condition of orthotomy (§ 8)

$$(1)\quad \lambda\tan a+\mu\tan b+\nu\tan c = 6V\cos r\sqrt{(\sec a\sec b\sec c)}$$
$$= (a\lambda\alpha'+b\mu\beta'+c\nu\gamma')\sqrt{(\sec a\sec b\sec c)}.$$

The centre moves on the dirigent (F),

$$(2)\quad la^2\lambda^2+mb^2\mu^2+nc^2\nu^2=0.$$

The equation to the sphero-cyclide, as the envelope of (1), subject to the condition (2), is

$$\frac{1}{la^2}\{a\alpha'\sqrt{(\sec a\sec b\sec c)}-\tan a\}^2$$
$$+\frac{1}{mb^2}\{b\beta'\sqrt{(\sec a\sec b\sec c)}-\tan b\}^2$$
$$+\frac{1}{nc^2}\{c\gamma'\sqrt{(\sec a\sec b\sec c)}-\tan c\}^2=0.$$

If we revert to the primitive triangle of reference ABC, it is written

$$\Sigma\frac{1}{l}\{a\alpha+b\beta\cos c+c\gamma\cos b-6V\sec a\sqrt{(\cos a\cos b\cos c)}\}^2=0$$

14. If U, V, W denote, as in § 11, the J_1, J_2, J_3 circles, the equation to the sphero-cyclide takes the form

$$\frac{U^2}{l}+\frac{V^2}{m}+\frac{W^2}{n}=0.$$

If this form be compared with the tangential equation of (F),

$$\frac{p^2}{l}+\frac{q^2}{m}+\frac{r^2}{n}=0,$$

as in § 11, $\qquad p:q:r::U:V:W.$

The equation to the sphero-cyclide was anticipated from Dr. Casey's theorem.

15. To express the three focal sphero-conics F_1, F_2, F_3 in terms of their confocal F.

Being given $\qquad F\equiv\frac{p^2}{l}+\frac{q^2}{m}+\frac{r^2}{n}=0,$

if ABC be the triangle of reference, to determine the coefficients, if OBC be the new triangle, in the assumed equation

$$F_1\equiv\frac{P^2}{l_1}+\frac{q^2}{n_1}+\frac{r^2}{m_1}=0.$$

P denotes the perpendicular from O $(\alpha\cos A=\beta\cos B=\gamma\cos C)$ on any tangent arc, so that

$$P^2\Sigma(\tan^2 A+2\tan B\tan C\cos a)=(p\tan A+q\tan B+r\tan C)^2$$
$$=\tan A\tan B\tan C\,(p^2\tan A\cos a+q^2\tan B\cos b+r^2\tan C\cos c)$$
$$-\Sigma(p^2\sin^2 A-2qr\cos A\sin B\sin C)\sec A\sec B\sec C.$$

Make this substitution, and denote by μ, as in § 9, the ratio

$$\Sigma(\tan^2 A+2\tan B\tan C\cos a)=\mu^2\tan^2 A\tan^2 B\tan^2 C,$$

$$F_1\equiv\frac{1}{l_1\mu^2}\cot A\cot B\cot C\,(p^2\tan A\cos a+q^2\tan B\cos b+r^2\tan C\cos c)$$
$$-\frac{6V}{l_1\mu^2}\cot A\cot B\cot C+\frac{q^2}{n_1}+\frac{r^2}{m_1}.$$

Since F and F_1 are doubly confocal conics, they must be identical, if the constant term be omitted. Write θ for $\frac{1}{\mu^2}\cot A\cot B\cot C$, and equate coefficients.

$$\frac{1}{l}=\frac{\theta}{l_1}\tan A\cos a,\quad \frac{1}{m}=\frac{\theta}{l_1}\tan B\cos b+\frac{1}{n_1};$$

$$\frac{1}{n}=\frac{\theta}{l_1}\tan C\cos c+\frac{1}{m_1}.$$

Substitute this value for θ in F_1,

$$F_1\equiv\frac{p^2}{l}+\frac{q^2}{m}+\frac{r^2}{n}-\frac{6V}{l}\cot A\sec a=0.$$

F_2, F_3 have similar forms.

16. The following identity connects the several (J) forms of § 12:

$$\begin{aligned}&\tan A\cos a\,\{a\alpha+b\beta\cos c+c\gamma\cos b-6V\sec a\sqrt{(\cos a\cos b\cos c)}\}^2\\&+\tan B\cos b\,\{a\alpha\cos c+b\beta+c\gamma\cos a-6V\sec b\sqrt{(\cos a\cos b\cos c)}\}^2\\&+\tan C\cos c\,\{a\alpha\cos b+b\beta\cos a+c\gamma-6V\sec c\sqrt{(\cos a\cos b\cos c)}\}^2\\&=\cos^2 a\cos^2 b\cos^2 c\tan A\tan B\tan C\\&\qquad\times\{\alpha\tan a+\beta\tan b+\gamma\tan c-6V\sqrt{(\sec a\sec b\sec c}\}^2.\end{aligned}$$

The proof depends upon identities of the type

$$\tan A+\tan B\cos c+\tan C\cos b=\cos b\cos c\tan A\tan B\tan C.$$

Hence it may be shown that the equation to the sphero-cyclide may be obtained from any other pair of circles (J_1) and dirigent conics (F_1).

When referred to tangential coordinates, OBC being the triangle considered,

$$(F_1)\qquad \frac{p^2}{l_1}+\frac{q^2}{n_1}+\frac{r^2}{m_1}=0.$$

By Prof. Casey's theorem, cited in § 11, II., the equation is deduced to the sphero-cyclide

$$\frac{J_1^2}{l_1}+\frac{J_2^2}{n_1}+\frac{J_3^2}{m_1}=0.$$

But, from this article,

$$\frac{J^2}{t}=\tan A\cos a\,J_1^2+\tan B\cos b\,J_2^2+\tan C\cos c\,J_3^2,$$

if $$\frac{1}{t}=\cos^2 a\cos^2 b\cos^2 c\tan A\tan B\tan C.$$

If this value of J^2 be substituted, and the result compared with

former equation $$\frac{J_1^2}{l}+\frac{J_2^2}{m}+\frac{J_3^2}{n}=0,$$

it is seen that

$$\frac{1}{l}=\frac{t}{l_1}\tan A\cos a,\qquad \frac{1}{m}=\frac{t}{l_1}\tan B\cos b+\frac{1}{n_1},$$

$$\frac{1}{n}=\frac{t}{l_1}\tan C\cos c+\frac{1}{m_1}.$$

These relations are those given in § 15.

17. If a spherical quadrilateral be circumscribed about a circle of inversion and its corresponding dirigent conic, the other three confocal dirigent conics pass through the three quartets of opposite intersections. (Casey on "Cyclides," § 124.)

The tangential equations to such a (J) circle and (F) conic (§ 12)

are (J) $p^2\tan A\cos a+q^2\tan B\cos b+r^2\tan C\cos c=0,$

$$(F)\qquad \frac{p^2}{l}+\frac{q^2}{m}+\frac{r^2}{n}=0.$$

The triangle of reference is constituted by the three vertices of the quadrangle of intersection, as before. Whence

$$\frac{p^2}{l}:\frac{q^2}{m}:\frac{r^2}{n}::m\tan B\cos b-n\tan C\cos c$$
$$:n\tan C\cos c-l\tan A\cos a$$
$$:l\tan A\cos a-m\tan B\cos b.$$

The spherical quadrilateral, thus constituted, is defined by the linear

equations $$a\alpha p\pm b\beta q\pm c\gamma r=0.$$

Two points of intersection, as well as their two antipodal points, lie on the arc BC, which passes through two vertices of the quadrangle

of intersection, $$\alpha=0,\quad b^2\beta^2q^2=c^2\gamma^2r^2.$$

Since the line-equation to F_1 (by § 15) is

$$(F_1)\quad \frac{p^2}{l}+\frac{q^2}{m}+\frac{r^2}{n}-\frac{1}{6Vl\tan A\cos a}\Sigma\,(a^2p^2-2bcqr\cos A)=0,$$

the transformed point-equation is

(F_1)

$$a^2\cos A\,(\mu\nu-\cos^2 A)-\frac{b\beta^2}{cn}\cos B\cos C\sin A\,(n\tan C\cos c-l\cos a\tan A)$$

$$-\frac{c\gamma^2}{bm}\cos B\cos C\sin A\,(m\tan B\cos b-l\cos a\tan A)$$

$$+2a\beta\cos B\,(\sin^2 A-\sin^2 C)+2a\gamma\cos C\,(\sin^2 A-\sin^2 B)=0,$$

if $$(1-\mu)\,b^2m=(1-\nu)\,c^2n=6Vl\tan A\cos a.$$

When $a=0$

$$mb^2\beta^2\,(n\tan C\cos c-l\cos a\tan A)=nc^2\gamma^2\,(l\cos a\tan A-m\cos b\tan B),$$

or $$b^2\beta^2q^2=c^2\gamma^2r^2.$$

The arc BC therefore meets the conic (F_1) in the preceding points.

Similarly, the other two conics (F_2), (F_3) may be shown to pass through the other intersections of the quadrilateral.

18. To find the equation to the sphero-cyclide in two-point coordinates, the origin being the centre of a dirigent focal conic.

The coordinates in Gudermann's system represent tangents of arcs, so that x, y; a, b represent

$$\tan x,\ \tan y;\quad \tan a,\ \tan b;\quad \sec^2 r=1+x^2+y^2.$$

The equation to a (J) circle, whose centre is (f, g) is

$$(J)\quad \cos\delta\sqrt{(1+f^2+g^2)}\sqrt{(1+x^2+y^2)}=1+fx+gy,$$

or $$1+fx+gy=\cos\delta\sec R\sec r=t\sec r,$$

where t denotes the secant of the tangent arc drawn from the origin.

The equation to the generating circle, whose centre is (α, β) is

$$1+\alpha x+\beta y=T\sec r.$$

The condition of orthotomy (§ 8) gives the relation

$$1+\alpha f+\beta g=tT.$$

For the generating circle, when T is eliminated,

$$t\,(1+\alpha x+\beta y)=(1+\alpha f+\beta g)\sec r\ \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

For the dirigent conic

$$(F)\quad \frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}=1\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The circle (1) varies, subject to the condition (2); hence

$$(f\sec r - tx)\,d\alpha + (g\sec r - ty)\,d\beta = 0,$$

$$\frac{\alpha\,d\alpha}{a^2} + \frac{\beta\,d\beta}{b^2} = 0.$$

The required equation to the sphero-cyclide is

$$(t-\sec r)^2 = a^2(f\sec r - tx)^2 + b^2(g\sec r - ty)^2.$$

When rationalised,

$$\{(a^2f^2+b^2g^2-1)\sec^2 r + t^2(a^2x^2+b^2y^2-1)\}^2 = 4t^2\sec^2 r\,(a^2fx+b^2gy-1)^2.$$

If this be combined with the imaginary great circle $\sec^2 r = 0$,

$$(a^2x^2+b^2y^2-1)^2 = 0 \quad \text{or} \quad \{(a^2-b^2)\,x^2-(1+b^2)+b^2(1+x^2+y^2)\}^2 = 0.$$

The sphero-cyclide has two double cyclic arcs, which are the single cyclic arcs of the polar or complementary conic of (F) the focal conic. The line through the nodes is the polar of (f, g) with respect to this polar conic $(a^2x^2+b^2y^2=1)$.

The formulæ of quadric transformation from the tangential equation of (F) $(a^2\xi^2+b^2\eta^2=1)$ are seen to be

$$\xi : \eta : 1 = f\sec r - tx : g\sec r - ty : t - \sec r,$$

where $$t = \cos\delta\,\sqrt{(1+f^2+g^2)}.$$

19. If the four (F) conics are given, to determine the four corresponding (J) circles.

The equation to the sphero-cyclide may take other three forms of the above type,

$$(t_1-\sec r)^2 = a_1^2(f_1\sec r - t_1x)^2 + b_1^2(g_1\sec r - t_1y)^2.$$

For the confocal conics,

$$\frac{1+a^2}{1+b^2} = \frac{1+a_1^2}{1+b_1^2} = \frac{1+a_2^2}{1+b_2^2} = \frac{1+a_3^2}{1+b_3^2} = 1+\gamma^2; \quad [\gamma = \tan(CS)];$$

C, S being a common centre and focus of the confocal conics.

By equating coefficients in the identical forms, when developed,

$$\left.\begin{aligned}
\frac{1}{t}(a^2f^2+b^2g^2-1+a^2t^2) &= \frac{1}{t_1}(a_1^2f_1^2+b_1^2g_1^2-1+a_1^2t_1^2)\\
\frac{1}{t}(a^2f^2+b^2g^2-1+b^2t^2) &= \frac{1}{t_1}(a_1^2f_1^2+b_1^2g_1^2-1+b_1^2t_1^2)\\
\frac{1}{t}(a^2f^2+b^2g^2-1-t^2) &= \frac{1}{t_1}(a_1^2f_1^2+b_1^2g_1^2-1-t_1^2)
\end{aligned}\right\} \quad \ldots\ldots(1),$$

$$a^2f = a_1^2f_1 = a_2^2f_2 = a_3^2f_3 = \lambda \text{ suppose,}$$
$$b^2g = b_1^2g_1 = b_2^2g_2 = b_3^2g_3 = \mu.$$

The line through the nodes $(a^2fx+b^2gy = 1)$ is fixed and seen to be the polar of the centre of each (J) circle with respect to the polar conic of its corresponding (F) conic.

$$(a^2+1)\,t = (a_1^2+1)\,t_1 = \dots = \nu \text{ suppose,}$$
$$(b^2+1)\,t = (b_1^2+1)\,t_1 = \dots = \frac{\nu}{1+\gamma^2}.$$

(1) may be written

$$\frac{1}{t}\left(\frac{\lambda^2}{a^2}+\frac{\mu^2}{b^2}-1\right)+a^2t = \frac{1}{t_1}\left(\frac{\lambda^2}{a_1^2}+\frac{\mu^2}{b_1^2}-1\right)+a_1^2t_1.$$

This is simplified to the condition of orthotomy,

$$\frac{\lambda^2}{a^2a_1^2}+\frac{\mu^2}{b^2b_1^2}+1 = tt_1, \quad\text{or}\quad ff_1+gg_1+1 = tt_1.$$

By symmetry, $\qquad ff_2+gg_2+1 = tt_2.$

By subtraction,

$$\frac{\lambda f}{a_1^2a_2^2}+\frac{\mu g}{b_1^2b_2^2} = 1, \quad\text{or}\quad \frac{\lambda^2}{a^2a_1^2a_2^2}+\frac{\mu^2}{b^2b_1^2b_2^2} = 1.$$

By symmetry, $\qquad \dfrac{\lambda^2}{a^2a_1^2a_3^2}+\dfrac{\mu^2}{b^2b_1^2b_3^2} = 1.$

Hence $\qquad \lambda^2\gamma^2 = a^2a_1^2a_2^2a_3^2,\quad \dfrac{\mu^2\gamma^2}{1+\gamma^2} = -\,b^2b_1^2b_2^2b_3^2.$

Since $\qquad \dfrac{\lambda^2}{a^2a_1^2}+\dfrac{\mu^2}{b^2b_1^2}+1 = tt_1 = \dfrac{\nu^2}{(a^2+1)(a_1^2+1)},$

$$\nu^2(1+\gamma^2) = (a^2+1)(a_1^2+1)(a_2^2+1)(a_3^2+1).$$

Thus the radii (δ) and the centres (f, g) of the (J) circles have been determined in terms of the axes of the (F) sphero-conics.

Cor.—If the centres of the (J) circles are collinear, the foci of the sphero-cyclide are also collinear. In this case, three, and not four, pairs of (F) focal conics and (J) circles are necessary; if $g = g_1 = g_2 = g_3 = 0$, $\lambda^2 = a^2a_1^2a_2^2$, $\nu^2 = (a^2+1)(a_1^2+1)(a_2^2+1)$.

20. To determine the two double and four single foci of the sphero-cyclide, and its equivalent class-octavic.

Its equation is taken from § 19, when m is written for $a^2f^2+b^2g^2-1$,

$$\{m\,(x^2+y^2+1)+t^2\,(a^2x^2+b^2y^2-1)\}^2 = 4t^2\,(x^2+y^2+1)(a^2fx+b^2gy-1)^2.$$

The equivalent class-equation is found by combining it with an arbitrary tangent $(x\xi+y\eta-1)$; the resulting binary quartic must have two equal roots, in which case $(I_4)^3-27\,(I_6)^2=0$, as in § 10.

The resulting binary quartic is

$$\{[m+a^2t^2+(m-t^2)\,\xi^2]\,x^2+2\,(m-t^2)\,\xi\eta\,xy+[m+b^2t^2+(m-t^2)\,\eta^2]\,y^2\}^2$$
$$=4t^2\,[(\xi^2+1)\,x^2+2\xi\eta xy+(\eta^2+1)\,y^2]\,[(a^2f-\xi)\,x+(b^2g-\eta)\,y]^2.$$

Then the system of invariants, given in § 10, has the following values:

$$\begin{aligned}A&=(m+a^2t^2)(m+b^2t^2)+(m-t^2)\,[(m+a^2t^2)\,\eta^2+(m+b^2t^2)\,\xi^2]\\&=t^4\,(a^2+1)\,(b^2+1)+(t^4-mt^2)\,(a^2\xi^2+b^2\eta^2-1)\\&\qquad+[t^4+mt^2\,(a^2+b^2-1)+m^2]\,(\xi^2+\eta^2+1),\\B&=\xi^2+\eta^2+1.\end{aligned}$$

In determining the foci, B and all other terms, which involve $(\xi^2+\eta^2+1)$ as a factor, are neglected, since the foci are obtained by the intersections of common tangents to the quartic and this imaginary great circle.

$$\begin{aligned}\frac{C}{4t^2}&=(a^2f-\xi)^2+(b^2g-\eta)^2+(a^2f\eta-b^2g\xi)^2\\&=(a^4f^2+b^4g^2+1)\,(\xi^2+\eta^2+1)-(a^2f\xi+b^2g\eta-1)^2,\\\frac{D}{4t^2}&=(m+b^2t^2)(a^2f-\xi)^2+(m+a^2t^2)(b^2g-\eta)^2+(m-t^2)(a^2f\eta-b^2g\xi)^2\\&=t^2\,[(b^2+1)(a^2f-\xi)^2+(a^2+1)(b^2g-\eta)^2]+(t^2-m)(a^2f\xi+b^2g\eta+1)^2\\&\qquad-(t^2-m)\,(a^4f^2+b^4g^2+1)(\xi^2+\eta^2+1),\\\frac{E}{4t^2}&=(\xi^2+1)\,[m+b^2t^2+(m-t^2)\,\eta^2]-2\xi^2\eta^2\,(m-t^2)\\&\qquad+(\eta^2+1)\,[m+a^2t^2+(m-t^2)\,\xi^2]\\&=-t^2\,(a^2\xi^2+b^2\eta^2-1)+[2m+(a^2+b^2-1)\,t^2]\,(\xi^2+\eta^2+1).\end{aligned}$$

If these values be substituted in the expression for $(I_4)^3-27\,(I_6)^2$ in § 10, the equivalent class octavic may be obtained.

The foci are found by rejecting B, and therefore the last three terms. For the double foci,

$$E^2=t^4\,(a^2\xi^2+b^2\eta^2-1)^2.$$

They are therefore the single foci of (F) the focal conic, from which the sphero-cyclide was generated.

The four single foci are those of the class-quartic

$$\left(A+\frac{C}{4}\right)^2-DE.$$

Cor.—The two double foci unite in a quadruple focus, when $a = b$, or (F) becomes a circle. This sphero-cyclide is called by Dr. Casey a sphero-Cartesian.

21. The sphero-cyclide has in all 28 foci, real and imaginary.

Prof. Cayley remarks that this quartic has two nodes, and, besides, touches the imaginary circle $S\,(x^2+y^2+1=0)$ in four points. The number of its class is thus $4 \,.\, 3 - 2 \,.\, 2$, or 8; and the number of common tangents to the quartic and the circle S would thus be $2 \,.\, 8$ or 16; but among these are included the four lines touching along the points of contact, each twice; the number of common tangents is thus $16-2 \,.\, 4$, or 8.

These eight lines intersect in $28 \left(=\dfrac{8 \,.\, 7}{1 \,.\, 2}\right)$ points, which are the foci of this quartic.

II. *On Sphero-cyclides with Collinear Foci.*

22. Their equations have been deduced in § 19, Cor., from the general form; but they are here also investigated, when a focus of the focal conic is the origin, and the constants determined in terms of the directrices.

There are only three (F) focal sphero-conics,

$$(F_1)\quad \lambda\,(x^2+y^2)-(x-a)^2=0, \qquad (F_2)\quad \mu\,(x^2+y^2)-(x-b)^2=0,$$

$$(F_3)\quad \nu\,(x^2+y^2)-(x-c)^2=0.$$

There are three corresponding coorthotomic circles of inversion,

$$(J_1)\ 1+fx=\rho_1 \sec r, \quad (J_2)\ 1+gx=\rho_2 \sec r, \quad (J_3)\ 1+hx=\rho_3 \sec r.$$

Let the generating circle for the first pair be

$$1+\alpha x+\beta y=\gamma\,(1+x^2+y^2)^{\frac{1}{2}}=\gamma \sec r.$$

The condition of orthotomy (§ 8) is

$$1+\alpha f=\rho_1 \gamma.$$

The envelope is required of the variable circle

$$\rho_1\,(1+\alpha x+\beta y)=(1+\alpha f)\sec r,$$

subject to the dirigent condition, that its centre moves on the conic,

$$\lambda\,(\alpha^2+\beta^2)-(\alpha-a)^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots..(F_1).$$

The equation to the envelope is found, as in § 19, to be

$$\{(1+af)\sec r-\rho_1(1+ax)\}^2+a^2\rho_1^2y^2=\lambda\,(\sec r-\rho_1)^2.$$

But, since the dirigent conics are doubly confocal, S, H being the common foci,

$$\frac{1}{2a}(\lambda-1+a^2) = \frac{1}{2b}(\mu-1+b^2) = \frac{1}{2c}(\nu-1+c^2) = d = -\cot SH.$$

The preceding equation to the sphero-cyclide may be written

$$\{a(\rho_1^2+f^2+1)+2f-2d\}\sec^2 r$$
$$+2\rho_1\{2d-(1+af)x-f-a\}\sec r+2\rho_1^2(x-d) = 0.$$

The quadrantal polars of the origin $(1=0)$, and of the other common focus $(x=d)$, are double cyclic arcs of the sphero-cyclide.

The line of nodes $[(1+af)x+f+a = 2d]$ is the polar of the centre of (J) with respect to the polar conic of (F_1),

$$(ax+1)^2+a^2\left(1-\frac{1}{\lambda}\right)y^2 = \lambda.$$

23. Two other forms may be written, in which ρ_2, g, b; ρ_3, h, c take the place of ρ_1, f, a. It is proposed to obtain thereby another form of the quartic, in which the coefficients shall be functions of a, b, c, the tangents of the distances of the directrices from that common focus, which is the origin of coordinates.

Equate the coefficients in the preceding and the identical quartic

$$[(1+bg)^2+b^2\rho_2^2-\mu]\sec^2 r-2\rho_2[(bg+1)+(bg+1)bx-\mu]\sec r$$
$$+2b\rho_2^2(x-d) = 0,$$

$$\frac{1}{\sqrt{K}} = \frac{af+1}{\rho_1} = \frac{bg+1}{\rho_2}, \qquad \frac{af+1-\lambda}{a\rho_1} = \frac{bg+1-\mu}{b\rho_2} = \frac{H}{\sqrt{K}},$$

$$\frac{a^2\rho_1^2+(af+1)^2-\lambda}{a\rho_1^2} = \frac{b^2\rho_2^2+(bg+1)^2-\mu}{b\rho_2^2}.$$

The symbols H, K are introduced for subsequent use.

These relations may be combined, so as to express ρ_1^2 in terms of f:

$$\frac{\rho_1}{\rho_2} = \frac{(a-b)(af+1)+b\lambda}{a\mu},$$

$$\rho_1^2(a-b)-\frac{a-b}{ab}(af+1)^2-\frac{\lambda}{a}+\frac{1}{a^2b\mu}[(a-b)(af+1)+b\lambda]^2 = 0.$$

After reduction,

$$\mu\rho_1^2+f^2(ab-1-2ad)-2f(a-b)+ab-1-2bd = 0 \quad \text{......(1).}$$

By symmetry,

$$\nu\rho_1^2+f^2(ac-1-2ad)-2f(a-c)+ac-1-2cd=0\quad\ldots\ldots\ldots(2).$$

The symbols λ, μ, ν are retained, for convenience, as known functions of a, b, c, and d.

Subtract (2) from (1), and reject the factor $(b-c)$.

$$(af+1)^2-\lambda-a\rho_1^2(b+c-2d)=0,$$

or

$$(af+1)^2+a^2\rho_1^2-\lambda=a\rho_1^2(a+b+c-2d).$$

Multiply (1) by c, and (2) by b; subtract, and reject the factor $(b-c)$.

$$(1+bc)\rho_1^2=(1+af)^2+\lambda f^2.$$

By eliminating ρ_1^2, there results a quadratic, which indicates two (J_1) circles,

$$f^2[-abc-a+b+c+2d(ab+ac-1)-4ad^2]$$
$$+2f(-bc+ab+ac-1-2ad)-abc-a+b+c+2bcd=0.$$

For brevity, write this quadratic

$$Af^2+2Bf+C=0.$$

The following relations connect the coefficients:

$$2Bd=A-C,\quad B^2-AC=\lambda\mu\nu.$$

Whence

$$Af+B=\sqrt{(\lambda\mu\nu)},$$

$$A(af+1)=A-aB+a\sqrt{(\lambda\mu\nu)}=\lambda(b+c-2d)+a\sqrt{(\lambda\mu\nu)},$$

$$\frac{A}{a}(1-\lambda)-B=\frac{\lambda}{a}(b+c-2d-A)=\lambda+\lambda(b-2d)(c-2d)$$
$$=\lambda+\frac{\lambda}{bc}(1-\mu)(1-\nu).$$

We have the proportion above stated

$$\frac{1}{a}\left(1-\frac{\lambda}{af+1}\right)=\frac{1}{b}\left(1-\frac{\mu}{bg+1}\right)=H.$$

Substitute the preceding value of $af+1$, and a similar value of $bg+1$,

$$H=\left[\frac{\lambda}{bc}(1-\mu)(1-\nu)+\lambda+\sqrt{(\lambda\mu\nu)}\right][\lambda(b+c-2d)+a\sqrt{(\lambda\mu\nu)}]^{-1}$$
$$=\left[\frac{\mu}{ac}(1-\nu)(1-\lambda)+\mu+\sqrt{(\lambda\mu\nu)}\right][\mu(c+a-2d)+b\sqrt{(\lambda\mu\nu)}]^{-1}.$$

Dividendo, $=\left\{\frac{1-\nu}{c}\left[\frac{\lambda}{b}(1-\mu)-\frac{\mu}{a}(1-\lambda)\right]+\lambda-\mu\right\}$

$$\times\left\{(a-b)\sqrt{(\lambda\mu\nu)}+\frac{\mu}{c-a}(\nu-\lambda)-\frac{\lambda}{b-c}(\mu-\nu)\right\}^{-1}$$

$$=\{-(a-2d)(b-2d)(c-2d)-a-b-c+4d\}$$

$$\times\left\{\sqrt{(\lambda\mu\nu)}-1+\frac{1}{2d}(a-2d)(b-2d)(c-2d)-\frac{abc}{2d}\right\}^{-1},$$

which is a symmetrical function.

By the aid of this proportion, the coefficients may be expressed.

From the former proportions,

$$\frac{1}{\rho_1^2}(af+1)^2=\frac{1}{\rho_2^2}(bg+1)^2,$$

$$(af+1)^2-\lambda=a\rho_1^2(b+c-2d),\quad\text{and}\quad\frac{1}{a}\left(1-\frac{\lambda}{af+1}\right)=H.$$

Hence $K=\rho_1^2(af+1)^{-2}=\frac{1}{a}(b+c-2d)^{-1}[1-\lambda(af+1)^{-2}]$

$$=\frac{1}{a}(b+c-2d)^{-1}\left[1-\frac{1}{\lambda}(aH-1)^2\right],$$

by symmetry,

$$=\frac{1}{b}(c+a-2d)^{-1}\left[1-\frac{1}{\mu}(bH-1)^2\right],$$

dividendo,

$$=\left[\left(\frac{b^2}{\mu}-\frac{a^2}{\lambda}\right)H^2-2\left(\frac{b}{\mu}-\frac{a}{\lambda}\right)H+\frac{1}{\mu}-\frac{1}{\lambda}\right](a-b)^{-1}(c-2d)^{-1}$$

$$=\frac{1}{\lambda\mu}(c-2d)^{-1}[-(2abd+a+b)H^2+2(ab+1)H+2d-a-b],$$

by symmetry,

$$=\frac{1}{\mu\nu}(a-2d)^{-1}[-(2bcd+b+c)H^2+2(bc+1)H+2d-b-c],$$

dividendo,

$$=\frac{a-c}{\mu}[-(2bd+1)H^2+2bH-1][\lambda(c-2d)-\nu(a-2d)]^{-1}$$

$$=\frac{1}{\mu}[(2bd+1)H^2-2bH+1]\left[1+\frac{1}{ac}(1-\lambda)(1-\nu)\right]^{-1},$$

by symmetry,

$$=\frac{1}{\nu}\left[(2cd+1)H^2-2cH+1\right]\left[1+\frac{1}{ab}(1-\lambda)(1-\mu)\right]^{-1},$$

dividendo,

$$=(-2dH^2+2H)\left\{\frac{1}{a}(1-\lambda)\left[1+\frac{1}{bc}(1-\mu)(1-\nu)\right]+b+c-2d\right\}^{-1}$$

$$=2H(-dH+1)\left\{\frac{1}{abc}(1-\lambda)(1-\mu)(1-\nu)+\frac{1}{a}(1-\lambda)+\frac{1}{b}(1-\mu)\right.$$
$$\left.+\frac{1}{c}(1-\nu)+2d\right\}^{-1}.$$

The equation to the sphero-cyclide may therefore be thus expressed symmetrically in terms of the tangents of the distances of the directrices of the focal conics:

$$\frac{1}{2}(a+b+c-2d)\sec^2 r-\frac{1}{\sqrt{K}}(x+H)\sec r+x-d=0.$$

Cor. 1.—If $a+b+c=2d$, the satellite-conic degenerates into the cyclic arcs.

Cor. 2.—If $K=0$, the sphero-cyclide degenerates into the line of nodes twice repeated, and the imaginary great circle.

Cor. 3.—If $K=\infty$, or $(a-2d)(b-2d)(c-2d)+a+b+c-4d$, the sphero-cyclide degenerates into two coincident conics, but retains the same cyclic arcs.

III. *On Sphero-Cartesians.*

24. To generate a Sphero-Cartesian by Laguerre's method.

There are three concentric dirigent circles,

$$x^2+y^2=a_1^2\ \ldots\ldots\ (F_1),\quad x^2+y^2=a_2^2\ \ldots\ldots\ (F_2),\quad x^2+y^2=a_3^2\ \ldots\ldots\ (F_3).$$

And to these there correspond three co-orthotomic circles of inversion,

$$1+f_1x=t_1\sec r\ \ldots\ldots\ (J_1),\qquad 1+f_2x=t_2\sec r\ \ldots\ldots\ (J_2),$$
$$1+f_3x=t_3\sec r\ \ldots\ldots\ (J_3),$$

where f_1, f_2, f_3 denote the coordinates of their centres, and t_1, t_2, t_3 the secants of the touching arcs drawn from the origin.

The equation to the cyclide is found, as in § 18, or deduced from it, as the envelope of a variable circle, which cuts the (J) circles

orthogonally,

$$\left(f_1^2+t_1^2-\frac{1}{a_1^2}\right)\sec^2 r+2t_1\left(f_1x-\frac{1}{a_1^2}\right)\sec r-t_1^2\left(1+\frac{1}{a_1^2}\right)=0.$$

Similar forms involve the constants f_2, a_2, t_2; f_3, a_3, t_3.

By equating the coefficients of like terms in the equivalent forms,

$$f_1a_1^2=f_2a_2^2=f_3a_3^2=a_1a_2a_3,$$

$$t_1(a_1^2+1)=t_2(a_2^2+1)=t_3(a_3^2+1)=\sqrt{\{(a_1^2+1)(a_2^2+1)(a_3^2+1)\}},$$

$$\begin{aligned}(a_1^2+1)\{(f_1^2+t_1^2)a_1^2-1\} &= (a_2^2+1)\{(f_2^2+t_2^2)a_2^2-1\}\\ &= (a_3^2+1)\{(f_3^2+t_3^2)a_3^2-1\}\\ &= 2a_1^2a_2^2a_3^2+a_2^2a_3^2+a_3^2a_1^2+a_1^2a_2^2-1.\end{aligned}$$

The sphero-Cartesian can now be expressed in terms of the radii of the (F) dirigent circles

$$\begin{gathered}(2a_1^2a_2^2a_3^2+a_2^2a_3^2+a_3^2a_1^2+a_1^2a_2^2-1)\sec^2 r-(a_1^2+1)(a_2^2+1)(a_3^2+1)\\ +2\sqrt{\{(a_1^2+1)(a_2^2+1)(a_3^2+1)\}}\,(a_1a_2a_3x-1)\sec r.\end{gathered}$$

25. To express the radii of the (J) circles of inversion in terms of those of the (F) dirigent circles. (Casey on "Cyclides," § 244.)

Let δ_1, δ_2, δ_3 denote the radii of the (J) circles:

$$\sec^2\delta_1=\frac{1+f_1^2}{t_1^2}=\left(1+\frac{a_2^2a_3^2}{a_1^2}\right)(a_1^2+1)(a_2^2+1)^{-1}(a_3^2+1)^{-1},$$

$$\tan^2\delta_1=\frac{1}{a_1^2}\cdot\frac{a_2^2-a_1^2}{1+a_2^2}\cdot\frac{a_3^2-a_1^2}{1+a_3^2}.$$

This may be also expressed in terms of the distances ρ_1, ρ_2, ρ_3 of the centres of the (J) circles

$$\tan^2\delta_1=\frac{f_1-f_3}{1+f_1f_3}\cdot\frac{f_1-f_2}{1+f_1f_2}=\tan(\rho_1-\rho_2)\tan(\rho_1-\rho_3).$$

IV. *On the Singularities of Sphero-cyclides.*

26. There is in all cases a pair of nodes, which may be real crunodes, or imaginary nodes; not acnodes. Thus sphero-cyclides are discriminated from other spherical binodal quartics.

When certain mutual relations exist between the parameters, which

enter into their equations, there may be another node, crunode, or acnode, which unite in a cusp.

Moreover, the two nodes in the line of nodes may unite in a tacnode; and the two nodes (which are imaginary) may coalesce with a third (acnode) to form a triple point, of Salmon's special form 5°. (*Higher Plane Curves*, § 243.) See below, § 30.

Lastly, these two singularities, the tacnode and the triple point, may (in a special case) coalesce, and form a compound singularity, called a tacnode-cusp, of Salmon's special form 4°.

27. To determine the mutual relation which subsists between the parameters in the equation to a family of sphero-cyclides, which have the same line of nodes and the same double cyclic arcs, when the sphero-cyclides are trinodal.

This relation will be drawn as a first discriminating curve (D_1).

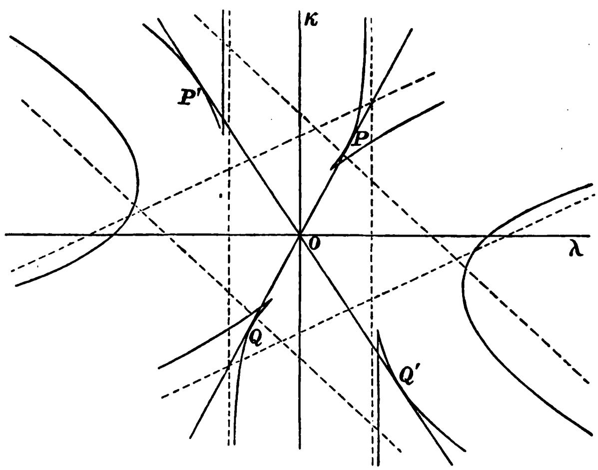

Fig. 3.

Let this equation take the form

$$\phi \equiv \kappa\,(1+dx)-(1+mx+py)\sec r+\lambda\sec^2 r=0,$$

where κ, λ are parameters, $\sec^2 r=1+x^2+y^2$ in Gudermann's system, $(1=0,\ 1+dx=0)$ denote the cyclic arcs, and $(1+mx+py=0)$ the line of nodes.

At a singular point,

$$\frac{d\phi}{dx}=0,\quad \frac{d\phi}{dy}=0.$$

For the sphero-cyclide,

$$\kappa d = m \sec r + x(1+mx+py)\cos r + 2\lambda x,$$

$$0 = p\sec r + y(1+mx+py)\cos r + 2\lambda y;$$

whence $\qquad \kappa(2+dx) = \sec r + (1+mx+py)\cos r + 2\lambda,$

and $\qquad \kappa dy = (my-px)\sec r,$

$$\kappa(2+dx) = \left(1-\frac{p}{y}\right)\sec r.$$

All the nodes in the family lie on the sphero-conic

$$\frac{my}{p}\left(x+\frac{2}{d}-\frac{1}{m}\right) = x^2+\frac{2x}{d}-1.$$

The locus of (κ, λ) may be drawn by points for successive values of x from $+\infty$ to $-\infty$. This curve (D_1) has a Newtonian centre, since the expressions for κ, λ both contain $\sec r$ or $(1+x^2+y^2)^{\frac{1}{2}}$.

It has six asymptotes $\qquad \lambda \pm p = 0,$

when $\qquad x = \dfrac{1}{m}-\dfrac{2}{d}.$

$$\kappa d - 2\lambda x \pm m(1+x^2)^{\frac{1}{2}} \pm x(1+mx)(1+x^2)^{-\frac{1}{2}} = 0,$$

where x has two values from the quadratic

$$x^2-\frac{2x}{d}+1 = 0.$$

It has a pair of points

$$\kappa = \frac{1}{md}(m^2+p^2)^{\frac{1}{2}}, \quad \lambda = -(m^2+p^2)^{\frac{1}{2}},$$

which correspond to the infinite values of x, y $(my = px)$. It has four pairs of cusps.

The line of nodes $(1+mx+py=0)$ will touch the sphero-conic

$$\kappa(1+dx)-\lambda(1+x^2+y^2) = 0,$$

if $\qquad d^2p^2\kappa^2+4(m^2+p^2-dm)\kappa\lambda = 4(m^2+p^2+1)\lambda^2.$

The two lines thus defined, as functions of κ and λ, both touch and cut the curve (D_1), as shown in Fig. 3. See remarks on PQ in § 28.

28. To determine the relation between the parameters, when sphero-cyclides with collinear foci are trinodal. (Curve D_1.)

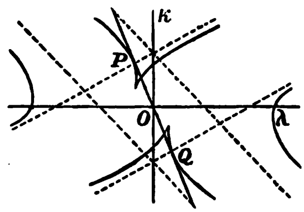

FIG. 4.

In § 27, $p = 0$; and the equation to a family of such quartics is

$$\kappa\,(1+dx) = (1+mx)\sec r + \lambda \sec^2 r.$$

The sphero-conic (§ 27), on which all the third nodes lie, is resolved into two great circles

$$\text{(I.)}\ y = 0, \qquad \text{(II.)}\ x = \frac{1}{m} - \frac{2}{d}.$$

The Curve (D_1) in this case consists of two parts, according as

$$\text{(I.)}\ y = 0, \quad \text{or} \quad \text{(II.)}\ 1+mx+2\lambda\sec r = 0.$$

I. When $y = 0$, κ, λ have the following values:

$$\kappa\,(dx^2+2x-d) = (x-m)\,(1+x^2)^{\frac{1}{2}},$$

$$\lambda\,(dx^2+2x-d)\,(1+x^2)^{\frac{1}{2}} + mdx^3+2mx^2+x+m-d = 0.$$

Two pairs of linear asymptotes are given by the equation

$$\kappa\,(1+dx) = (1+mx)\,(1+x^2)^{\frac{1}{2}} + \lambda\,(1+x^2),$$

where x has two values given by the quadratic

$$dx^2+2x-d = 0.$$

A pair of cusps is determined by the value of x, which satisfies both the conditions $\qquad \dfrac{d\kappa}{dx} = 0, \quad \dfrac{d\lambda}{dx} = 0.$

By taking the logarithmic differential of κ,

$$\frac{2dx+2}{dx^2+2x-d} = \frac{1}{x-m} + \frac{x}{1+x^2}.$$

That is, $\qquad 2\,(x+m)+d\,(mx^3-3x^2+3mx-1) = 0 \quad \ldots\ldots\ldots\ldots(\text{A}).$

Its discriminant is $\quad 4\,(d^2+1)\,\{(m^2-1)\,d+2m\}^2 = 0.$

Let $d = \dfrac{2m}{1-m^2}$, or $\tan\delta = \tan 2\mu$, that is, let the distance from

the origin of the arc of nodes be half that of the cyclic arc

$$\kappa d\,(x-m)\left(x+\frac{1}{m}\right) = (x-m)\,(1+x^2)^{\frac{1}{2}},$$

$$2\lambda m\,(x-m)\left(x+\frac{1}{m}\right)(1+x^2)^{\frac{1}{2}}+(x-m)\,(2m^2x^2+2mx+m^2+1) = 0.$$

There is but one solution applicable, viz., $x = m$, $y = 0$; and consequently it belongs to the subsequent Case II. The curve drawn with the values of κ, λ, when the factor $(x-m)$ is withdrawn, is inapplicable, and does not constitute a part of the Curve (D_1).

The cubic (A) has only one real solution; it has no equal roots, when $d = \frac{2m}{1-m^2}$, unless $m = 1$, and $d = \infty$. Consequently, there is only one pair of cusps for each sphero-cyclide of this family.

If the line of nodes $(1+mx = 0)$ touch the sphero-conic

$$\kappa\,(1+dx) = \lambda \sec^2 r,$$

then

$$m\kappa\,(m-d) = (m^2+1)\,\lambda.$$

If this ratio $\kappa : \lambda$ be substituted in their preceding values as functions of x,

$$(mx+1)^2\,\{x\,(1+md)+m-d\} = 0 \quad \text{..................(B).}$$

Hence this line PQ $(\kappa:\lambda)$, in Figs. 3, 4, 5, both touches and cuts Curve (D_1).

To their point of contact there corresponds a sphero-cyclide with a triple point (see § 26); to their point of intersection, a quartic with a tacnode and crunode; moreover, this line PQ discriminates quartics with real crunodes from those with imaginary nodes.

II. For the second portion of Curve (D_1), when

$$x = \frac{1}{m}-\frac{2}{d}, \quad 1+mx+2\lambda \sec r = 0,$$

there results the hyperbola

$$\frac{d}{m^2}\,\kappa\lambda = \frac{1}{d}-\frac{1}{m}.$$

For all values of κ, λ thus correlated, the quartic degenerates into two coincident circles, codiametral with the (x) arc,

$$1+mx+2\lambda\,(1+x^2+y^2)^{\frac{1}{2}} = 0.$$

Cor.—For the sphero-Cartesian, the conditions for a third node or other singularity are derived, when $d = 0$. (Fig. 5.)

[29. To find the condition for a tacnode-cusp in a sphero-cyclide.

In this case the tangent at the cusp in the Curve (D_1) coincides with the line PQ, which therefore meets (D_1) in three coincident points; the triple point unites with a cusp in the corresponding quartic.

The values of x in (B) are identical, if $2md = m^2-1$, or $\tan\delta+\cot 2\mu = 0$, or $\frac{\pi}{2}+\delta = 2\mu$ (p. 133).

This value of d satisfies (A), the condition of a cusp:

$$(mx+1)\{(m^2+3)x^2-4mx+(1+3m^2)\} = 0.$$

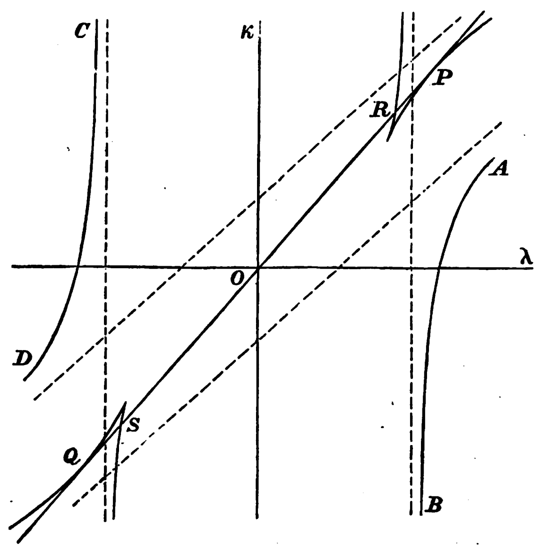

Fig. 5.

The two sphero-Cartesians

$$\sec^2 r\pm(1\pm x)\sec r = 2$$

have each a tacnode-cusp, when $x = \mp 1$ respectively.]

30. To determine all the varieties of sphero-cyclides with collinear foci by the aid of the first discriminating curve. (Figs. 4, 5.)

Let the line POQ be first considered: for all values of the parameters κ, λ, for points above this line in the first quadrant, and below this line in the third quadrant, all such quartics have two crunodes; for points on the other sides of this line, the two characteristic nodes are imaginary. At the points of intersection R, S, the quartics have a

tacnode and crunode; at the points of contact P, Q, a triple point formed by the union of an acnode with two imaginary nodes.*

To the cusp in this Curve (D_1) there corresponds a cusp in the sphero-cyclide, besides the two imaginary nodes; for points on (D_1) on either side of the cusp, crunodes and acnodes correspond.

For points (κ, λ) on Curve (D_1) above R, the Sphero-Cartesians are tricrunodal; and for points beyond P, acnodal with two imaginary nodes.

For points (κ, λ) on either side of Curve (D_1), the companion-curves become bipartite, or unipartite with folia.

The outer lines in the second and fourth quadrants AB, CD are bounding lines: *i.e.*, for points (κ, λ) thereon, the corresponding sphero-cyclides shrink to points; and for exterior points, none correspond.

V. *On the Points of Undulation in Sphero-cyclides.*

31. To determine the mutual relation between the parameters, which enter into their equations, when sphero-cyclides have points of undulation, or the second discriminating curve (D_2).

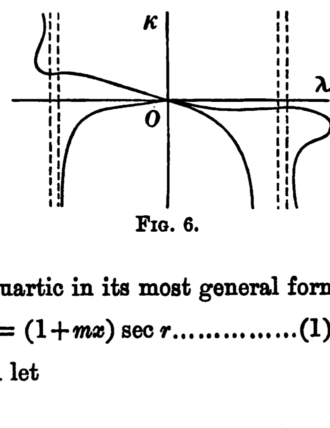

FIG. 6.

At a folium-point, or point of undulation, $\dfrac{d^2y}{dx^2} = 0$, $\dfrac{d^3y}{dx^3} = 0$.

Let these tests be applied to this quartic in its most general form (§ 27)

$$\lambda \sec^2 r - \kappa (1 + dx + ey) = (1 + mx) \sec r \quad \text{..............(1)}.$$

Differentiate thrice successively, and let

$$p = \frac{dy}{dx}.$$

$$2\lambda (x + py) - \kappa (d + ep) = m \sec r + (x + py)(1 + mx) \cos r \quad \text{......(2)},$$

$$2\lambda (1 + p^2)$$
$$= 2m(x + py) \cos r + (1 + p^2)(1 + mx) \cos r - (x + py)^2 (1 + mx) \cos^3 r \quad \text{...(3)},$$

$$0 = m (1 + p^2) \cos r - m (x + py)^2 \cos^3 r - (1 + p^2)(x + py)(1 + mx) \cos^3 r$$
$$+ (1 + mx)(x + py)^3 \cos^5 r.$$

---

* [This triple point discriminates sphero-cyclides from other binodal quartics. In the quartic $\kappa (1 + dx) = (1 + mx)\sqrt{(1 + x^2 - y^2)} + \lambda (1 + x^2 - y^2)$, the triple point for the same critical values of κ and λ, as in the text, has one real, and two real coincident branches, caused by the union of a crunode with two real nodes.]

This last condition may be resolved into two factors

$$(1+p^2)\sec^2 r-(x+yp)^2, \text{ and } m\sec^2 r-(x+py)(1+mx) \quad \ldots(4).$$

The former factor is rejected, as it should be from its form

$$1+p^2+(px-y)^2,$$

since $$\sec^2 r = 1+x^2+y^2.$$

The second factor (4) gives the real condition: when substituted in (2) and (3),

$$2\lambda(x+py)-\kappa(d+ep) = 2m\sec r \quad \ldots\ldots\ldots\ldots(2),$$

$$2\lambda(1+p^2)\sec r = m(x+py)+(1+p^2)(1+mx) \ldots\ldots\ldots(3).$$

The elimination of κ and λ from (1), (2), and (3) gives another condition,

$$(d+ep)\sec^2 r\,[m(x+py)+(1+p^2)(1+mx)]$$
$$-2(x+py)(1+dx+ey)[m(x+py)+(1+p^2)(1+mx)]$$
$$+2(1+p^2)\sec^2 r\,[2m(1+dx+ey)-(d+ep)(1+mx)] = 0.$$

By the aid of (4) this eliminant may take the form

$$2m(1+dx+ey)[(1+p^2)\sec^2 r-(x+py)^2]$$
$$= (d+ep)\sec^2 r\;[(1+p^2)(1+mx)-m(x+py)],$$

or a still simpler form by the aid of (4),

$$2m(1+dx+ey)-(d+ep)(1+mx) = 0 \quad \ldots\ldots\ldots\ldots(5).$$

By eliminating p from (4) and (5), the locus of the points of undulation in a family of sphero-cyclides is seen to be a sphero-conic.

$$2m-d+mdx+2mey = \frac{e}{y}[m(1+y^2)-x] \quad \ldots\ldots\ldots\ldots(6).$$

By giving y successive values from $+\infty$ to $-\infty$, single values of x, and therefrom of κ, λ, so that the required discriminating curve (D_2) is drawn. The curve (D_2) has the origin for a Newtonian centre, since κ, λ are determined as factors of $\sec r$ or $(1+x^2+y^2)^{\frac{1}{2}}$.

It has an asymptote, when

$$d+ep = 0;$$

and $$1+dx+ey = 0, \qquad ex+(m-d)\,y = me.$$

For these values $\kappa = \infty$, and λ is known from (3).

It has a second asymptote, also parallel to the (κ) axis, when

$$m\,dy+e=0, \quad x=\infty, \quad \text{and} \quad ep=d;$$

for these values

$$\kappa=\infty, \quad 2\lambda\,(d^2+e^2)=m\,(d^2+2e^2).$$

There is a node at the origin, corresponding to the value ($1+mx=0$), since it yields two values of y.

Only half of the curve is drawn in Fig. 6.

32. To determine the mutual relation between the parameters, when sphero-cyclides with collinear foci have points of undulation. Curve (D_2).

In this case $e=0$ in § 31; and the locus of the points of undulation in such a family is a great circle, co-diametral with the (y) arc,

$$2m-d+mdx=0 \quad \ldots\ldots\ldots\ldots\ldots(6).$$

The limiting values of y are ∞ and 0; negative values of y give the same negative value of p from (4), and therefore the same points, since κ and λ are functions of

$$(x+yp) \text{ and } (1+p^2).$$

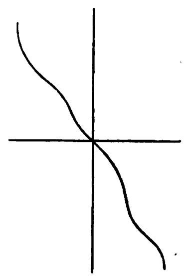

Fig. 7.

When $y=\infty$, $p=\infty$; and $\kappa=0=\lambda$ determines the origin.

When $y=0$, $p=\infty$; and κ, λ have these finite values,

$$(3) \quad 2\lambda\,(1+x^2)^{\frac{1}{2}}=1+mx; \qquad (2) \quad 2\lambda m\,\frac{1+x^2}{1+mx}-d\kappa=2m\,(1+x^2)^{\frac{1}{2}};$$

and the curve (D_2) terminates abruptly. The origin is its centre.

If $d=0$, the form fails, since (2) and (3) are incompatible, and by (6) $m=0$. Sphero-Cartesians have therefore no point of undulation.

The points of undulation are limiting forms of folia or depressions, characterised by two points of inflexion. Accordingly, for values of (κ, λ) on one side or the other of the Curves (D_2), in Figs. 6, 7, the companion sphero-cyclides have two points of inflexion or none.

VI. *On the Sphero-du-cyclide.*

33. This is the polar or complementary curve of the sphero-cyclide; its modes of generation and the forms of its equations are derived at

once by dualising (§ 36). It is a class-quartic with two bitangents, real or imaginary, and therefore of the eighth order.

The sphero-du-cyclide has two double foci, which are the single foci of the focal conic of the sphero-cyclide.

It is generated in four ways, as the envelope of a variable small circle, whose concentric great circle rolls on a dirigent sphero-conic (F), and which small circle is so related to a fixed small circle (J), that their common tangents are quadrants in length.

The four dirigent conics (F) are doubly concyclic; their two common cyclic arcs are double in the sphero-du-cyclide, thence generated.

34. Def.—The *quadranto-circle* of a spherical triangle ABC has the common orthocentre of ABC and its polar triangle $A'B'C'$ for its own pole, and is such that two corresponding sides BC, $B'C'$ of the primitive and polar triangles meet each other and the quadranto-circle in points A'', A'', which are 90° distant from A.

The great circles, which are concentric with the four (J) small circles, taken by threes, constitute four spherical triangles, which are self-conjugate with respect both to the (F) conics and the (J) circles. Each triad has the fourth great circle for its quadranto-circle.

The other properties of the du-cyclide are in like manner readily derived from those of the sphero-cyclide.

35. The transformations of § 11 have their counter-part, in transforming from a sphero-conic to a sphero-du-cyclide.

I. The dual formulæ (1), (2) become

$$\sin r = \cos(r-p)\sin\delta, \quad \tan r = \frac{\cot p}{\operatorname{cosec} p \operatorname{cosec}\delta - 1} \quad \text{.........(1)},$$

and of anallagmatic inversion

$$\frac{1-\tan\frac{p_1}{2}}{1+\tan\frac{p_1}{2}}\cdot\frac{1-\tan\frac{p_2}{2}}{1+\tan\frac{p_2}{2}} = \left\{\frac{1-\tan\frac{\delta}{2}}{1+\tan\frac{\delta}{2}}\right\}^2 \quad \text{..............(2)}.$$

II. If U, V, W denote the fixed circles J_1, J_2, J_3, which have the property that their common tangent arcs are quadrants in length, and if the dirigent conic (F) be denoted by the point equation

$$(a, b, c, f, g, h \between \alpha, \beta, \gamma)^2 = 0,$$

then the sphero-du-cyclide, thence generated (§ 33), has an equation of the same form $\quad (a, b, c, f, g, h \between U, V, W)^2 = 0.$

The symbol U has the dual form of that given in § 11,

$$U = \cos AP - \cos \delta_1.$$

AP now denotes the inclination of an arbitrary circle (ξ, η) to the great circle (l, m) concentric with (J_1), whose radius is $\frac{\pi}{2} - \delta_1$, and

equation $$U \equiv \frac{l\xi + m\eta + 1}{\sqrt{(l^2+m^2+1)}\sqrt{(\xi^2+\eta^2+1)}} - \cos \delta_1 = 0.$$

36. Universally in Analytical Spherics, the formulæ of any curve are applicable to its dual or polar curve, by writing p', q', r', the line-coordinates of a great circle referred to $A'B'C'$, the polar triangle, for α, β, γ, the point-coordinates of the dual point or spherical centre referred to ABC, the primitive triangle.

In Gudermann's system, where the triangles ABC, $A'B'C'$ coincide, being tri-quadrantal, $-\xi, -\eta$ must be substituted for x, y, in dualising.

All the equations in this memoir are equally applicable to this dual curve, when the point and line-coordinates are thus transformed.

VII. *On Doubly Bi-confocal Sphero-cyclides.*

37. Dr. Casey has applied the methods of sphero-conics to sphero-cyclides by the transformation given in § 11. ("On Cyclides," §§ 270—273, 308—311.)

I will apply it to transform the general equation of confocal sphero-

conics $$lp^2 + mq^2 + nr^2 + \lambda\,(apP + bqQ + crR) = 0,$$

and to prove that all sphero-cyclides, which have the same pair of double foci, but different single foci, pass through four concircular points, which lie on a small circle, whose pole or spherical centre is the orthocentre of the triangle of reference, or centre of the J circle from which it is generated.*

We have to interpret

$$a^2U^2 + b^2V^2 + c^2W^2 - 2bcVW\cos a - 2caWU\cos b - 2abUV\cos c,$$

* [Doubly bi-confocal bicircular quartics and cyclides do not intersect. The function $(apP + bqQ + crR)$, when quadrically transformed, gives the centre of J or the orthocentre of the centres of J_1, J_2, J_3 as a point-circle

$$\Sigma\,(a^2\alpha^2 \cos^2 A - 2bc\beta\gamma \cos A \cos B \cos C),$$

not situate on the curve. A system of doubly bi-confocal Cartesians and Cartesian cyclides is expressed by the relation $S \propto D$, where S is a circle or sphere, and D the distance of a point on the locus from the centre of the (F) focal circle or sphere.]

where, by § 11, II., in Dr. Casey's notation,

$$p \propto U \propto \cos at - \cos aP \propto \mu \sec a - a',$$

$$q \propto V \propto \mu \sec b - \beta', \quad r \propto W \propto \mu \sec c - \gamma'.$$

For brevity, μ stands for $\sqrt{(\cos a \cos b \cos c)}$,

$$6va' = a\alpha + b\beta \cos c + c\gamma \cos b.$$

Hence $\quad P = ap - bq \cos C - cr \cos B \propto \mu\nu \cos B \cos C - 6v\alpha,$

$$Q \propto \mu\nu \cos C \cos A - 6v\beta, \quad R \propto \mu\nu \cos A \cos B - 6v\gamma.$$

Here also, for brevity, ν denotes $\tan a \tan b \tan c$, $6v = ab \sin C$.

$$\begin{aligned} apP + bqQ + crR \propto\ & a(\mu \sec a - a')(\mu\nu \cos B \cos C - 6v\alpha) \\ & + b(\mu \sec b - \beta')(\mu\nu \cos C \cos A - 6v\beta) \\ & + c(\mu \sec c - \gamma')(\mu\nu \cos A \cos B - 6v\gamma) \end{aligned}$$

$$\begin{aligned} \propto\ & (6v)^2 + abc(\tan a \cos B \cos C + \tan b \cos C \cos A + \tan c \cos A \cos B) \\ & - 12v\sqrt{(\cos a \cos b \cos c)}\,(\alpha \tan a + \beta \tan b + \gamma \tan c). \end{aligned}$$

This is the equation to a small circle, concentric with the J circle of § 13.

$$abc\,\Sigma(\tan a \cos B \cos C) = (6v)^2 - abc \tan a \tan b \tan c \cos A \cos B \cos C.$$

The reductions used here depend on the identity

$$\tan b \cos C + \tan c \cos B = \tan a\,(1 - \tan b \tan c \cos B \cos C).$$

The transformed equation to this family of sphero-cyclides is

$$\begin{aligned} & l(\mu \sec a - a')^2 + m(\mu \sec b - \beta')^2 + n(\mu \sec c - \gamma')^2 \\ & + \lambda\{2(6v)^2 + abc \tan a \tan b \tan c \cos A \cos B \cos C \\ & \qquad + 12v\mu(\alpha \tan a + \beta \tan b + \gamma \tan c)\}^2. \end{aligned}$$

The small circle intersects the sphero-cyclide in four points, and not in eight, unless the antipodal points be added, since, by expressing the constants in terms of $\Sigma \alpha \tan a$, the first line becomes a sphero-conic.

[38. Sphero-cyclides are cut by planes in four concircular points. As in § 37, any sphero-cyclide (§ 6)

$$a \sec^2 R + (bu_1 + v_1) \sec R + v_2 = 0$$

is intersected by any circle,

$$\sec R = lx + my + n = w,$$

in the same four points as the sphero-conic

$$aw_1^2 + (bu_1 + v_1)\, w_1 + v_2 = 0.]$$

Errata.—In § 4, *read* "The line of nodes in the sphero-cyclide is the polar of the centre of any (J) circle with respect to the polar conic of the corresponding (F) focal conic."

In § 6, Cor., *read* "If the cyclide has the imaginary circle at infinity as a cuspidal edge, or if $v_2 \equiv c\,(x^2+z^2)+dy^2$, the generated sphero-cyclide is called by Dr. Casey a Sphero-Cartesian."

Prof. Hart has already published his Memoir on the "Five Focal Quadrics of a Cyclide," in the *Messenger of Mathematics*, Vol. XIV., pp. 1—8. See § 5.

On the Limits of Multiple Integrals. *By* Hugh MacColl, B.A.

[*Read November 13th*, 1884.]

In my first paper on the "Calculus of Equivalent Statements" (*Proceedings*, Vol. IX., Nos. 124, 125), I showed how the limits of integration in a multiple integral might always be ascertained whenever we had enough data for the purpose. I now propose to show how the expression of the limits thus ascertained may often be simplified and reduced to a form more convenient for integration.

Definition.—When we have an elementary statement of the form $x_{m'.n}$ or $y_{m'.n}x_{r'.s}$ or $z_{m'.n}y_{r'.s}x_{u'.v}$, &c., presenting the *nearest* limits (or true limits of integration), we may, for brevity's sake, take any of these symbols to denote, *not* the statement itself, but the *integral* which has the limits of integration indicated by the statement.

Thus $y_{m'.n}x_{r'.s}$ will be a mere abbreviation for

$$\int_{y_n}^{y_m} dy \int_{x_s}^{x_r} dx.$$

From this definition we get the following self-evident rules:

RULE 1: $$x_{m'.n} = -x_{n'.m},$$

$$y_{m'.n}\, x_{r'.s} = -y_{n'.m}\, x_{r'.s} = y_{n'.m}\, x_{s'.r},$$

and so on.

RULE 2: $$x_{n'.n} = 0.$$

RULE 3: $$x_{m'.n} + x_{r'.s} = x_{m'.s} + x_{r'.n};$$

and, generally, in adding any number of integrals of the same function and variable, the superior limits may interchange their inferior limits in any order.

RULE 4: $$x_{m'.n} + x_{n'.r} = x_{m'.r},$$

and so on for any number of terms.

As an example of the application of these rules, take

$$y_{3'.4}\,(x_{3'.1} + x_{2'.4}) + (y_{3'.1} + y_{2'.4})\, x_{1'.2}.$$

Applying Rule 3 to the bracket integrals, the expression becomes

$$y_{3'.4}\,(x_{3'.4} + x_{2'.1}) + (y_{3'.4} + y_{2'.1})\, x_{1'.2} = y_{3'.4}\, x_{3'.4} + y_{2'.1}\, x_{1'.2};$$

for the two terms omitted cancel each other by Rule 1.

As another example, take

$$z_{1'.0}\, y_{1'.0}\, x_{2'.0} + z_{1'.0}\, y_{2'.0}\, x_{1'.2} + z_{2'.0}\, y_{1'.2}\, x_{1'.2}.$$

By Rule 4, we can put this into the form

$$z_{1'.0}\, y_{1'.0}\,(x_{2'.1} + x_{1'.0}) + z_{1'.0}\,(y_{2'.1} + y_{1'.0})\, x_{1'.2} + (z_{2'.1} + z_{1'.0})\, y_{1'.2}\, x_{1'.2}.$$

Multiplying out and cancelling (as in the preceding example) those terms which destroy each other by Rule 1, we get

$$z_{1'.0}\, y_{1'.0}\, x_{1'.0} - z_{1'.2}\, y_{1'.2}\, x_{1'.2}.$$

When the data from which we have to find the limits of integration involve *literal constants*, this circumstance often gives rise to different cases, the solution being different for each case.

When we have found the limits of the *variables*, we must then apply the same process to the *constants*, and determine their limits

also. In this way we shall separate the cases. Suppose, for example, the final result to be

$$Ua_1 + Va_{1'} + W,$$

in which U, V, and W are each expressions of the form

$$y_{m'.n}\,x_{r'.s} + y_{u'.v}\,x_{a'.\beta},$$

while a is a constant. Multiply the above final result by $a_1 + a_{1'}$, and we get

$$a_1(U+W) + a_{1'}(V+W).$$

This asserts that $U+W$ is the proper expression for the integration when $a > a_1$, but that $V+W$ is the proper expression when a is less than a_1.

Next, suppose the final expression to be

$$Ua_{1'.2'} + Va_2 + Wa_1 + Z.$$

Multiply by $(a_1+a_{1'})(a_2+a_{2'})$, and we get

$$a_{1.2}(V+W+Z) + a_{2'.1}(W+Z) + a_{1'.2}(V+Z) + a_{2'.1'}(U+Z).$$

This asserts that $V+W+Z$ is the proper expression for the integration when a_1 and a_2 are both inferior limits of a; that $W+Z$ is the proper expression when a is greater than a_1 and less than a_2; and so on.

The cases $a_{2'.1}$ and $a_{1'.2}$ in the above example are contradictory. When mere inspection of the table of limits is not sufficient to show which case must be expunged as impossible, we must have recourse to the formula $a_{m'.n} : p\,(a\ -a_n)$.

Since the transformations and simplifications, founded on Rules 1, 2, 3, 4 of this paper, are independent of any particular table of limits, it is evident that they may often be greatly facilitated by diagrams representing merely *rectangular* areas or volumes, without having recourse to curves or representations of curved surfaces. One or two simple examples will sufficiently illustrate my meaning. Take the expression

$$y_{2'.1}\,x_{1'.0} + 2y_{2'.1}\,x_{2'.1} + y_{1'.0}\,x_{2'.1}.$$

Fig. 1 makes it evident that this is identical with the simpler expression

$$y_{2'.1}\,x_{2.0} + y_{2'.0}\,x_{2'.1},$$

the elementary area containing the double *plus* representing the

middle term of the original expression. Suppose we had taken the

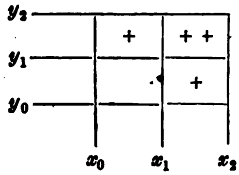

FIG. 1.

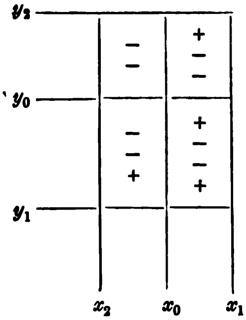

FIG. 2.

order of the limiting lines given in Fig. 2, we should have obtained our signs thus:

Observing the usual convention as to signs in regard to lines and areas, the first term $y_{2'.1}\,x_{1'.0}$ of the original expression directs us to insert a *plus* in each of the right-hand two elementary areas which make up the whole *positive* area $y_{2'.1}\,x_{1'.0}$. The second term $2y_{2'.1}\,x_{2'.1}$ directs us to insert *twice* a *minus* sign in each of the four elementary areas which make up the whole negative area $y_{2'.1}\,x_{2'.1}$. The third term $y_{1'.0}\,x_{2'.1}$ directs us to insert a *plus* sign in each of the two bottom elementary areas which make up the whole *positive* area $y_{1'.0}\,x_{2'.1}$. The whole collection of signs will then appear as in Fig. 2. Cancelling now the double sign $\pm$ wherever found, Fig. 2 will then assume the simpler form exhibited in Fig. 3, which makes it evident (as before) that the simplest expression for the area is the two-term expression

$$y_{2'.1}\,x_{2'.0}+y_{2'.0}\,x_{2'.1}\,.$$

It is also clear, from Fig. 3, that the expression may also be put into the form

$$2y_{2'.0}\,x_{2'.0}+y_{1'.0}\,x_{0'.2}+y_{2'.0}\,x_{0'.1},$$

which is not so clear from Fig. 1.

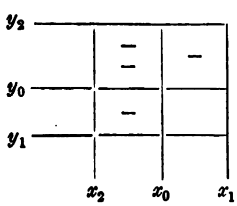

FIG. 3.

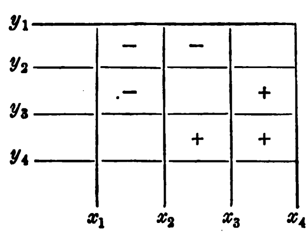

FIG. 4.

Take next the expression illustrated in Fig. 4. At first sight it

does not look as if this could be expressed in fewer than four terms; but place a double sign $\pm$ in the central blank elementary area, and it then becomes evident at once that the expression may be put into the simpler form

$$y_{2'.4}\,x_{4'.2}+y_{1'.3}\,x_{1'.3}.$$

Of course, all simplifications thus obtained by mere inspection of rectangular areas, or (without the aid of diagrams) by the preceding symbolical rules of reduction, hold good for all possible tables of limits, and therefore for limits the true representations of which would be *curves*, and not straight lines. It sometimes happens, however, when we come to actual integration, that, of two equivalent expressions, the one involves imaginary quantities, while the other does not. The final result, however, will be the same, whatever expression we take; for, in such cases, when imaginary quantities enter into our integration, they will always be found to vanish afterwards by cancelling.

In the case of three variables, as the elementary volumes which collectively represent the whole integration cannot generally be *all* exhibited to the eye, we cannot adopt the *plus* and *minus* system of marking which has just been illustrated in the case of areas. The following method is the best substitute that I can think of.

Let the symbol $m\cdot n\cdot r$ marked on the top block of a pile indicate that the m^{th}, n^{th}, and r^{th} blocks of that pile (counting upwards from the bottom) form parts of the integration, but that no other block of that pile (or column) forms part of it. A dot *above* a number asserts that the block indicated by that number is to be taken *positively*, while a dot *below* a number asserts that the block indicated by that number is to be taken *negatively*. For example, $\underset{\cdot}{\ddot{3}}\cdot\underset{\cdot\cdot}{\dddot{4}}\cdot\underset{\cdot}{\dot{5}}$ (which refers to blocks of the same column; see Fig. 5) asserts that the third block (counting upwards) is taken *twice* positively and *once* negatively; that the fourth block is taken *three* times positively and *twice* negatively; and that the fifth block is taken *once* positively and *once* negatively. Evidently, therefore, the above symbol may be replaced by $\dot{3}\cdot\dot{4}$, which asserts that the *third* and *fourth* blocks (shaded in Fig. 5) enter positively into the integration, the other blocks of the column being left out of account.

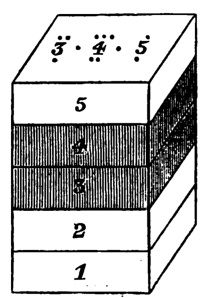

FIG. 5.

As an illustration, let it be required to simplify the expression

$$z_{3'.0}\,y_{3'.0}\,x_{1'.3}+z_{1'.0}\,y_{1'.3}\,x_{1'.3}+z_{3'.0}\,y_{1'.0}\,x_{3'.0}.$$

Considering upward, eastward, and northward as the positive directions, and their opposites as the negative, we shall obtain the result given in Fig. 6. Cancelling the double-signed (and therefore zero) volumes $\dot{1}\cdot\dot{2}$ and $\dot{1}$, we shall have left the positive column $1\cdot\dot{2}$ and the negative block 2, expressed

by $z_{3'.0}\,y_{1'.0}\,x_{1'.0} - z_{3'.1}\,y_{3'.1}\,x_{3'.1}$,

which is the utmost simplification of which the given expression is capable. The same result may, of course, be obtained, without the aid of a diagram, by the preceding rules, but, I think, less readily.

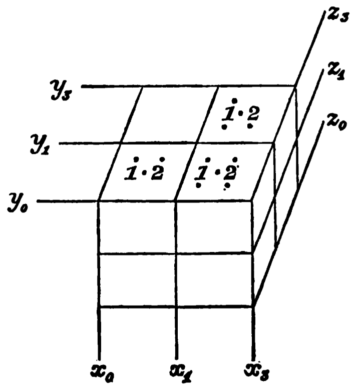

FIG. 6.

Note.—In evaluating multiple integrals of the form $z_{m'.n}\,y_{r'.s}\,x_{u'.v}$ and $\phi\,(x,\,y,\,z)\,z_{m'.n}\,y_{r'.s}\,x_{u'.v}$, the following notation would, I think, be convenient:

Let the symbols $\phi\,(x)\,x_{m'.n}$ and $x_{m'.n}\,\phi\,(x)$, which differ only in arrangement, have different meanings by virtue of this difference of arrangement. Let the former denote the value of the integral $\int dx\,\phi\,(x)$, when taken between the higher limit x_m and the lower limit x_n; and let the latter denote $\phi\,(x_m)-\phi\,(x_n)$.

For example, let $\int dx\,\phi\,(x) = \psi\,(x)$. According to the proposed convention, we should have

$$\phi\,(x)\,x_{m'.n} = x_{m'.n}\,\psi\,(x) = \psi\,(x_m)-\psi\,(x_n).$$

As a simple illustration, suppose we are required to evaluate $z_{1'.2}\,y_{1'.2}\,x_{1'.0}$ with reference to the accompanying table.

$z_1 = y$	$y_1 = x$	$x_1 = a$
$z_2 = c$	$y_2 = b$	

The full process would be as follows:

$$\begin{aligned} z_{1'.2}\,y_{1'.2}\,x_{1'.0} &= (y-c)\,y_{1'.2}\,x_{1'.0} \\ &= y_{1'.2}\,(\tfrac{1}{2}y^2-cy)\,x_{1'.0} \\ &= (\tfrac{1}{2}x^2-cx-\tfrac{1}{2}b^2+bc)\,x_{1'.0} \\ &= x_{1'.0}\,(\tfrac{1}{6}x^3-\tfrac{1}{2}cx^2-\tfrac{1}{2}b^2x+bcx) \\ &= \tfrac{1}{6}a^3-\tfrac{1}{2}a^2c-\tfrac{1}{2}ab^2+abc. \end{aligned}$$

The ordinary symbols of integration might thus be dispensed with, while the successive steps of the integration would be exhibited in their natural order and without a break.

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*On Quadric Transformations.** By Mr. W. SPOTTISWOODE, P.R.S.

A quantic is said to be transformed when the variables originally contained in it are replaced by others, and the transformation is called linear, or quadratic, or of a higher degree, according as the equations connecting the two sets of variables are of the first, or second, or of higher degrees. The transformations usually considered are linear, whereby, for example, a rational homogeneous function U of x, y, is transformed into another of the same degree in ξ, η, either by a substitution of the form

$$\xi : \eta = lx+my : l'x+m'y,$$

or by a substitution of the form

$$x : y = \alpha\xi+\beta\eta : \alpha'\xi+\beta'\eta.$$

The theory of linear transformations has, as is well known, been investigated by Boole, by Sylvester, by Cayley, and by many others; and notably by the last-mentioned mathematician in his classical "Memoirs on Quantics" (*Phil. Trans.*, 1856, *et seqq.*).

Tchirnhausen's transformation is an instance of one of a higher degree; and the substitution is of the first form, viz., it is as follows:

$$\xi : \eta = V : V',$$

where V, V' are homogeneous functions of x, y, of a given degree n. And, if U be the given function of x, y, which is to be the subject of transformation, the process is effected by eliminating x, y from the equations $$U=0, \quad \eta V-\xi V=0;$$

* Some papers by the late Mr. W. Spottiswoode, P.R.S., which had apparently been written with a view to their being brought before this Society, were, at the instance of the Council, submitted to Prof. Cayley for him to report upon the advisability of their being printed in the *Proceedings*. In accordance with Prof. Cayley's report, the following portions, which have been kindly edited by him, are printed here.

and the result, say $W = 0$, will clearly be of the same degree in ξ, η as was U in x, y. This done, we may calculate the derivatives (invariants and covariants) of W, and examine the relations between the derivatives in question and those of U, V, V', taken separately or in combination.

But, precisely as in linear transformations there are two forms, one in which ξ, η are expressed in terms of x, y, and the other in which x, y are expressed in terms of ξ, η (which forms may be termed complementary to one another); so also in higher transformations there are two forms, one expressed, as above, by Tchirnhausen's formula, and the other by the formula

$$x : y = \Upsilon : \Upsilon',$$

where Υ, Υ' are homogeneous functions of ξ, η, of a given degree. And, even although there be not at present any applications of the latter method whose utility is immediately apparent, it may still be a legitimate problem to calculate the derivatives of W, and to examine their relation to those of U, Υ, Υ'. This is the subject proposed in the present communication, wherein, however, I confine myself to quadratic transformations, and in fact to an elementary instance of such.

The question, in the case of a Tchirnhausen transformation, was noticed by Prof. Cayley in a paper entitled "An Example of the Higher Transformation of a Binary Form" (*Mathematische Annalen*, vol. IV., p. 359), and it was more fully investigated, especially with reference to quadratic transformations, by Gordan, in a paper, "Ueber die Invarianten binären Formen bei höheren Transformationen" (*Crelle*, vol. LXVII., p. 164). In connexion with the same subject, the following Memoirs may also be mentioned:—Hermite, "Sur quelques théorèmes d'algèbre, et la résolution de l'équation du quatrième degré" (*Comptes Rendus*, vol. XLVI., p. 961, 1858); also, Cayley, "On Tchirnhausen's Transformation" (*Crelle*, vol. LVIII., pp. 259–262, 263–269, 1858, and *Phil. Trans.*, 1862, pp. 566–578).

The case which we first propose to examine is the simplest, viz., the quadratic transformation of the quadric. Let the quadratic be

$$U = (a, b, c)(x, y)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and the equation of transformation, or the transformant,

$$x : y = (\alpha, \beta, \gamma)(\xi, \eta)^2 : (\alpha', \beta', \gamma')(\xi, \eta)^2, \;= V : V' \quad \ldots\ldots (2).$$

But, before proceeding to the actual transformation, it will be convenient to enumerate in detail the quadrics in x, y; as well as

in ξ, η; together with the several derivatives which will occur in the sequel.

Adopting the following notation,

$$D = ac - b^2,$$

$$\Delta = a\gamma - \beta^2,$$

$$2\Delta' = a\gamma' + a'\gamma - 2\beta\beta',$$

$$\Delta'' = a'\gamma' - \beta'^2,$$

$$\square = 4(\Delta\Delta'' - \Delta'^2),$$

$$K = a\Delta + 2b\Delta' + c\Delta'',$$

then D is the discriminant of U,

Δ „ „ V,

Δ'' „ „ V',

$2\Delta'$ is the intermediate of Δ, Δ'',

$\square$ is the resultant of V, V',

K is the intermediate of D, $\square$;

$\square$ is also the discriminant of a quantic U', to be hereafter particularised.

Further, putting $\left\| \begin{matrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{matrix} \right\| = A, B, C,$

we have the relations

$$\alpha A + \beta B + \gamma C = 0,$$

$$\alpha' A + \beta' B + \gamma' C = 0,$$

and $$4AC - B^2 = 4(\Delta\Delta'' - \Delta^2) = \square;$$

also, putting $\left\| \begin{matrix} c, & -2b, & a \\ \Delta, & 2\Delta', & \Delta'' \end{matrix} \right\| = P, Q, R,$

we have the relations $$cP - 2bQ + aR = 0,$$

$$\Delta P + 2\Delta' Q + \Delta'' R = 0,$$

and $$PR - Q^2 = D\square - K^2.$$

This being premised, if we write the transformant in the following form,

$$(ay - \alpha' x,\ \beta y - \beta' x,\ \gamma y - \gamma' x)(\xi, \eta)^2 = 0,$$

and then form its discriminant U' with respect to ξ, η, we shall find

$$U' = \Delta y^2 - 2\Delta' yx + \Delta'' x^2,$$

which is in fact the function, whose discriminant is $\frac{1}{4}\square$, mentioned above.

Following the course pursued by Cayley in his "Fifth Memoir on Quantics," and considering first the quantics in x, y, we have the two quadrics U, U'; their discriminants D, $\frac{1}{4}\square$; the connective of D, $\square$, viz., K; the resultant of the two quadrics, S; their Jacobian Θ; an intermediate Υ; and, lastly, the discriminant of the intermediate. Among these, the values of U, U', D, $\square$, K have been given above. Also it will be found that

$$S = 4\,(D\square - K^2),$$

$$\Theta = \tfrac{1}{2}\,(P,\ Q,\ R)\,(x,\ y)^2;$$

and, if the intermediate be

$$\Upsilon = (\lambda a + \mu\Delta'',\ \lambda b - \mu\Delta',\ \lambda c + \mu\Delta)\,(x,\ y)^2,$$

then its discriminant will be

$$(D,\ K,\ \tfrac{1}{4}\square\,\rangle\,(\lambda,\ \mu)^2.$$

If the two quadrics are harmonically related; that is, if their four roots are harmonically related; then

$$K = 0.$$

If they have a common factor, then

$$D\square - K^2 = PR - Q^2;$$

and the latter form expresses also the condition that the Jacobian shall be a perfect square, as was shown in the case of any two quadrics harmonically related, by Cayley in the Memoir above quoted.

Turning now to the quantics in ξ, η, we have the three quadrics,

viz., $$\begin{aligned} V &= (\alpha,\ \beta,\ \gamma)\,(\xi,\ \eta)^2, \\ V' &= (\alpha',\ \beta',\ \gamma')\,(\xi,\ \eta)^2, \\ V'' &= (\alpha y - \alpha' x,\ \beta y - \beta' x,\ \gamma y - \gamma' x)\,(\xi,\ \eta)^2, \end{aligned}$$

and, since the invariant of these, viz.,

$$\begin{vmatrix} \alpha, & \alpha', & \alpha y - \alpha' x \\ \beta, & \beta', & \beta y - \beta' x \\ \gamma, & \gamma', & \gamma y - \gamma' x \end{vmatrix} = 0,$$

vanishes identically, it follows that there is a syzygetic relation between V, V', V''; and therefore each quadric may be considered as an intermediate of the other two. The three quadrics are then said to be in involution; that is, the six roots of the three quadrics are in involution.

If to these three quadrics we add the Jacobian of V, V, viz.,

$$\Omega = (C, \ -B, \ A\between (\xi, \eta)^2,$$

we shall have four quadrics. Further, if the roots of these be represented by p, p'; q, q'; r, r'; s, s'; then the sets p, q, r, s; p', q', r', s' will be homographic, if

$$\begin{vmatrix} 1, & p, & p', & pp' \\ 1, & q, & q', & qq' \\ 1, & r, & r', & rr' \\ 1, & s, & s', & ss' \end{vmatrix} = 0.$$

which condition may be written also in the following form,

$$\begin{vmatrix} p-p', & 1, & p+p', & pp' \\ q-q', & 1, & q+q', & qq' \\ r-r', & 1, & r+r', & rr' \\ s-s', & 1, & s+s', & ss' \end{vmatrix} = 0.$$

But, multiplying the four lines of this determinant by α, α', $\alpha y - \alpha' x$, C, respectively, and substituting from the expressions for V, V', V'', Ω, the condition takes the form:

$$\begin{vmatrix} \sqrt{-\Delta}, & \alpha, & \beta, & \gamma \\ \sqrt{-\Delta''}, & \alpha', & \beta', & \gamma' \\ \sqrt{-U'}, & \alpha y - \alpha' x, & \beta y - \beta' x, & \gamma y - \gamma' x \\ \sqrt{-\Box}, & C, & -B, & A \end{vmatrix} = 0.$$

Then, writing for (line 3), $-y$ (line 1) $+ x$ (line 2) + (line 3), the determinant may be reduced to the following form:

$$(-y\sqrt{-\Delta} + x\sqrt{-\Delta''} + \sqrt{-U'}) \begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma \\ C, & -B, & A \end{vmatrix} = 0,$$

or $$(-y\sqrt{-\Delta} + x'\sqrt{-\Delta''} + \sqrt{-U'}) \Box = 0.$$

If the first factor vanishes, we may transpose the last term, and then ... th sides of the equation. We shall then obtain the following

expression:

$$y^2\Delta-2xy\sqrt{\Delta\Delta''}+x^2\Delta''=y^2\Delta-2xy\Delta'+x^2\Delta'',$$

whence also $\qquad \Delta\Delta''-\Delta'^2=\tfrac{1}{4}\,\square=0.$

Therefore, the condition that the roots of the four quadrics may form a homographic system is $\square=0$; that is, the quadrics V, V' must have a common factor.

To calculate the anharmonic ratio of V, V', V'', Ω. From the quadratic equations for p, q, r, s respectively, we have

$$\begin{aligned} p &= (-\beta+\sqrt{-\Delta}) && :\alpha, \\ q &= (-\beta'+\sqrt{-\Delta''}) && :\alpha', \\ r &= (-\beta y+\beta' x+\sqrt{-U'}) && :(\alpha y-\alpha' x), \\ s &= \tfrac{1}{2}\,(B+\sqrt{-\square}) && :C, \end{aligned}$$

and whence

$$\begin{aligned} q-r &= \{-\beta'(\alpha y-\alpha' x)+\alpha'(\beta y-\beta' x)+(\alpha y-\alpha' x)\sqrt{-\Delta''}-\alpha'\sqrt{-U'}\} \\ &\qquad\qquad :\alpha'(\alpha y-\alpha' x) \\ &= \{-Cy+(\alpha y-\alpha' x)\sqrt{-\Delta''}-\alpha'\sqrt{-U'}\}:\alpha'(\alpha y-\alpha' x), \\ r-p &= \{-\alpha(\beta y-\beta' x)+\beta(\alpha y-\alpha' x)-(\alpha y-\alpha' x)\sqrt{-\Delta}+\alpha\sqrt{-U'}\} \\ &\qquad\qquad :\alpha(\alpha y-\alpha' x) \\ &= \{\ \ Cx-(\alpha y-\alpha' x)\sqrt{-\Delta}+\alpha\sqrt{-U'}\}:\alpha(\alpha y-\alpha' x), \\ p-q &= \{C+\alpha'\sqrt{-\Delta}-\alpha\sqrt{-\Delta''}\}:\alpha\alpha', \\ p-s &= \{-2C\beta+2C\sqrt{-\Delta}-\alpha B-\alpha\sqrt{-\square}\}:2\alpha C \\ &= \{2\alpha\Delta'-2\alpha'\Delta+2C\sqrt{-\Delta}-\alpha\sqrt{-\square}\}:2\alpha C, \\ q-s &= \{-2C\beta'+2C\sqrt{-\Delta''}-\alpha' B-\alpha'\sqrt{-\square}\}:2\alpha' C \\ &= \{2\alpha\Delta''-2\alpha'\Delta'+2C\sqrt{-\Delta''}-\alpha'\sqrt{-\square}\}:2\alpha' C, \\ r-s &= \{-2C(\beta y-\beta' x)+2C\sqrt{-U'}-B(\alpha y-\alpha' x)-(\alpha y-\alpha' x)\sqrt{-\square}\} \\ &\qquad\qquad :2C(\alpha y-\alpha' x) \\ &= \{2(\alpha'\Delta'-\alpha\Delta'')y+2(\alpha\Delta'-\alpha'\Delta)x+2C\sqrt{-U}-(\alpha y-\alpha' x)\sqrt{-\square}\} \\ &\qquad\qquad :2C(\alpha y-\alpha' x); \end{aligned}$$

and the anharmonic ratio will be expressed by any of the following ratios:

$$(q-r)(p-s):(r-p)(q-s):(p-q)(r-s).$$

Each of these products has for its denominator the quantity

$$2aa'(ay-a'x),$$

which may therefore be discarded. Taking the first two of the above products, we shall have as the expression for the anharmonic ratio the following:

$$\{-Cy+(ay-a'x)\sqrt{-\Delta''}-a'\sqrt{-U'}\}$$
$$\times\{2a\Delta'-2a'\Delta+2C\sqrt{-\Delta}-a\sqrt{-\Box}\}$$
$$:\{Cx-(ay-a'x)\sqrt{-\Delta}+a\sqrt{-U'}\}$$
$$\times\{2a\Delta''-2a'\Delta'+2C\sqrt{-\Delta''}-a'\sqrt{-\Box}\}.$$

If the system be homographic, then, as was shown above, $\Box=0$, and,

$$\pm C \quad = a'\sqrt{-\Delta}-a\sqrt{-\Delta''},$$
$$\pm\sqrt{-U'}=y\sqrt{-\Delta}-x\sqrt{-\Delta''};$$

and, if we take C with the negative sign and $\sqrt{-U'}$ with the positive, we shall find that each of the factors of the anharmonic ratio will vanish, as they should, because the four quadrics have a common factor, *i.e.*, $p=q=r=s$.

If we give C the positive sign, retaining also the positive sign for $\sqrt{-U'}$, we shall find that

$$-Cy+(ay-a'x)\sqrt{-\Delta''}-a'\sqrt{-U'}=2y(a\sqrt{-\Delta''}-a'\sqrt{-\Delta}),$$
$$-Cx+(ay-a'x)\sqrt{-\Delta}\ -a\sqrt{-U'}=2x(a\sqrt{-\Delta''}-a'\sqrt{-\Delta}),$$
$$2(a\Delta'-a'\Delta+C\sqrt{-\Delta})-a\sqrt{-\Box}=4(a\Delta'-a'\Delta),$$
$$2(a\Delta''-a'\Delta'+C\sqrt{-\Delta''})-a'\sqrt{-\Box}=4(a\Delta''-a'\Delta').$$

Hence the anharmonic ratio of the roots not common to the four quadrics will be

$$=-y(a\Delta'-a'\Delta):x(a\Delta''-a'\Delta')$$
$$=y(\alpha',\beta',\gamma')(\beta,-\alpha)^2:x(\alpha,\beta,\gamma)(\beta',-\alpha')^2.$$

We now proceed to the transformed quantic itself. If we substitute in

$$U=(a,b,c)(x,y)^2,$$

by means of the transformant

$$x:y=(\alpha,\beta,\gamma)(\xi,\eta)^2:(\alpha',\beta',\gamma')(\xi,\eta)^2=V:V',$$

we shall obtain the result

$$W=(a,b,c)\{(\alpha,\beta,\gamma)(\xi,\eta)^2,\ (\alpha',\beta',\gamma')(\xi,\eta)^2\}^2,$$

and, if this be developed according to powers of ξ, η, we shall find

$$\begin{aligned} W = & \quad (a, b, c)(\alpha, \alpha')^2 & \xi^4 \\ & +4\ (a, b, c)(\alpha, \alpha')(\beta, \beta') & \xi^3\eta \\ & +2\{(a, b, c)(\alpha, \alpha')(\gamma, \gamma')+2(a, b, c)(\beta, \beta')^2\} & \xi^2\eta^2 \\ & +4\ (a, b, c)(\beta, \beta')(\gamma, \gamma') & \xi\eta^3 \\ & +\ \ (a, b, c)(\gamma, \gamma')^2 & \eta^4. \end{aligned}$$

It is proposed to calculate the invariants and covariants of W, and to express them in terms of those of U, V, V'.

Writing $$W = (\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e})(\xi, \eta)^4,$$

and putting

$$(a, b, c)(\beta, \beta')^2 = B, \quad (a, b)(\beta, \beta') = B_1, \quad (b, c)(\beta, \beta') = B_1',$$
$$a\Delta + 2b\Delta' + c\Delta'' = K,$$

we have
$$\begin{aligned} \mathrm{a} &= (a, b, c)(\alpha, \alpha')^2, \\ \mathrm{b} &= B_1\alpha + B_1'\alpha', \\ \mathrm{c} &= B + \tfrac{1}{3}K, \\ \mathrm{d} &= B_1\gamma + B_1'\gamma', \\ \mathrm{e} &= (a, b, c)(\gamma\gamma')^2. \end{aligned}$$

Now
$$\begin{aligned} \mathrm{ae} &= a^2(\Delta+\beta^2)^2 + 4b^2(\Delta+\beta^2)(\Delta''+\beta'^2) + c^2(\Delta''+\beta'^2)^2 \\ &\quad + 4ab(\Delta+\beta^2)(\Delta'+\beta\beta') + 4bc(\Delta''+\beta'^2)(\Delta'+\beta\beta') \\ &\quad + 2ac\{(\Delta'+\beta\beta')^2 - 2(\Delta+\beta^2)(\Delta''+\beta'^2)\} \\ &= \{a(\Delta+\beta^2) + 2\beta(\Delta'+\beta\beta') + c(\Delta''+\beta'^2)\}^2 \\ &\quad - 4(ac-b^2)\{(\Delta+\beta^2)(\Delta''+\beta'^2) - (\Delta'+\beta\beta')^2\} \\ &= (K+B)^2 - 4D(\Delta\Delta'' - \Delta'^2 + \Delta\beta'^2 - 2\Delta'\beta\beta' + \Delta''\beta^2); \\ -4\mathrm{bd} &= -4\{B_1^2(\Delta+\beta^2) + 2B_1B_1'(\Delta'+\beta\beta') + B_1'^2(\Delta''+\beta'^2)\} \\ &= -4\{\Delta B_1^2 + 2\Delta' B_1 B_1' + \Delta'' B_1'^2 + B^2\}; \\ 3\mathrm{c}^2 &= 3B^2 + 2BK + \tfrac{1}{3}K^2. \end{aligned}$$

Hence

$$\begin{aligned} \mathrm{ae} - 4\mathrm{bd} + 3\mathrm{c}^2 = \tfrac{4}{3}K^2 + 4BK - 4D(\Delta\Delta'' - \Delta'^2 + \Delta\beta'^2 - 2\Delta'\beta\beta' + \Delta''\beta^2) \\ - 4(\Delta B_1^2 + 2\Delta' B_1 B_1' + \Delta'' B_1'^2). \end{aligned}$$

But $\quad \Delta B_1^2+2\Delta' B_1 B_1'+\Delta'' B_1'^2 = \Delta\,(a^2\beta^2+2ab\,\beta\beta'+b^2\beta'^2)$

$$+2\Delta'\,(ab\,\beta^2+(ac+b^2)\,\beta\beta'+bc\beta'^2)+\Delta''\,(b^2\beta^2+2bc\,\beta\beta'+c^2\beta'^2)$$

$$=(a\Delta+2b\Delta'+c\Delta'')(a\beta^2+2b\,\beta\beta'+c\beta'^2)-(ac-b^2)(\Delta''\beta^2-2\Delta'\beta\beta'+\Delta\beta'^2)$$

$$= BK-D\,(\Delta\beta'^2-2\Delta'\beta\beta'+\Delta''\beta^2).$$

Hence $$\mathrm{ae}-4\mathrm{bd}+3\mathrm{c}^2 = \tfrac{4}{3}K^2-D\,\square.$$

We next come to the quarticovariant, which may be thus expressed:

$$(\mathrm{ac}-\mathrm{b}^2),\ 2\,(\mathrm{ad}-\mathrm{bc}),\ \mathrm{ae}+2\mathrm{bd}-3\mathrm{c}^2,\ 2\,(\mathrm{be}-\mathrm{cd}),\ (\mathrm{ce}-\mathrm{d}^2\big)\,(\xi,\eta)^4.$$

Now

$$\mathrm{ac}-\mathrm{b}^2 = (a\alpha^2+2b\alpha\alpha'+c\alpha'^2)\,(B+\tfrac{1}{3}K)-(B_1^2\alpha^2+2B_1B_1'\alpha\alpha'+B_1'^2\alpha'^2)$$

$$= D\,(\alpha\beta'-\alpha'\beta)^2+\tfrac{1}{3}K\mathrm{a}.$$

Next $\mathrm{ad}-\mathrm{bc}$; this is

$$= (a\alpha^2+2b\alpha\alpha'+c\alpha'^2)\,(B_1\gamma+B_1'\gamma')-(B_1\alpha+B_1'\alpha')\,(B+\tfrac{1}{3}K)$$

$$= aB_1\,(\Delta+\beta^2)\,\alpha+aB_1'\,(2\Delta'+2\beta\beta'-\alpha'\gamma)\,\alpha$$

$$+2bB_1\,(\Delta+\beta^2)\,\alpha'+2bB_1'\,(\Delta''+\beta'^2)\,\alpha$$

$$+\ cB_1\,(2\Delta'+2\beta\beta'-\alpha\gamma')\,\alpha'+cB_1'\,(\Delta''+\beta'^2)\,\alpha'$$

$$-\ \ B_1\,(B+\tfrac{1}{3}K)\,\alpha-B_1'\,(B+\tfrac{1}{3}K)\,\alpha'$$

$$= aB_1\,(\Delta+\beta^2)\,\alpha+2aB_1'\,(\Delta'+\beta\beta')\,\alpha-aB_1'\,(\Delta+\beta^2)\,\alpha'$$

$$+2bB_1\,(\Delta+\beta^2)\,\alpha'+2bB_1'\,(\Delta''+\beta'^2)\,\alpha$$

$$+2cB_1\,(\Delta'+\beta\beta')\,\alpha'+cB_1'\,(\Delta''+\beta'^2)\,\alpha'-cB_1\,(\Delta''+\beta'^2)\,\alpha$$

$$-B_1\,(B+\tfrac{1}{3}K)\,\alpha-B_1'\,(B+\tfrac{1}{3}K)\,\alpha'$$

$$= \{aB_1\,(\Delta+\beta^2)+2aB_1'\,(\Delta'+\beta\beta')+2bB_1'\,(\Delta''+\beta'^2)$$

$$-cB_1\,(\Delta''+\beta'^2)-B_1\,(B+\tfrac{1}{3}K)\}\,\alpha$$

$$+\{-aB_1'\,(\Delta+\beta^2)+2bB_1\,(\Delta+\beta^2)+2cB_1\,(\Delta'+\beta\beta')$$

$$+cB_1'\,(\Delta''+\beta'^2)-B_1'\,(B+\tfrac{1}{3}K)\}\,\alpha'.$$

But $$2bB_1'-cB_1 = bB_1'-D\beta,$$

$$2bB_1-aB_1' = bB_1-D\beta'.$$

Hence $$aB_1\beta^2+2aB_1'\beta\beta'+2bB_1'\beta'^2-cB_1\beta'^2-B_1B$$

$$= a\,(B_1\beta+B_1'\beta')\,\beta+B_1B_1'\beta'-D\beta\beta'^2-B_1B$$

$$= a\beta B+b\beta'B-B_1B = 0.$$

Hence the expression in question

$$= \{aB_1\Delta + 2aB_1'\Delta' + bB_1'\Delta'' - D\beta\Delta'' - \tfrac{1}{3}KB_1\}\,\alpha$$
$$+ \{cB_1'\Delta'' + 2cB_1\Delta' + bB_1\Delta - D\beta'\Delta - \tfrac{1}{3}KB_1'\}\,\alpha'$$
$$= \{(a^2\Delta + 2ab\Delta' + b^2\Delta'' - D\Delta'' - \tfrac{1}{3}Ka)\,\beta + (ab\Delta + 2ac\Delta' + bc\Delta'' - \tfrac{1}{3}Kb)\,\beta'\}\,\alpha$$
$$+ \{(bc\Delta'' + 2ac\Delta' + ab\Delta - \tfrac{1}{3}Kb)\,\beta + (c^2\Delta'' + 2bc\Delta' + b^2\Delta - D\Delta - \tfrac{1}{3}Kc)\,\beta'\}\,\alpha'$$
$$= 2\{(\tfrac{1}{3}aK - D\Delta'')\,\beta + (\tfrac{1}{3}bK + D\Delta')\,\beta'\}\,\alpha$$
$$+ 2\{\tfrac{1}{3}bK + D\Delta')\,\beta + (\tfrac{1}{3}cK - D\Delta)\,\beta\,\}\,\alpha'$$
$$= 2(\tfrac{1}{3}aK - D\Delta'',\ \tfrac{1}{3}bK + D\Delta',\ \tfrac{1}{3}cK - D\Delta)\,(\alpha, \alpha')\,(\beta, \beta')$$
$$= -2D\{\Delta\alpha'\beta' - \Delta\,(\alpha\beta' + \alpha'\beta) + \Delta''\alpha\beta\} + \tfrac{2}{3}K\mathrm{b},$$

or, finally, $\quad \mathrm{ad} - \mathrm{bc} = -2D\,(\alpha\beta' - \alpha'\beta)\,(\gamma\alpha' - \gamma'\alpha) + \tfrac{2}{3}Kb.$

Next $\quad \mathrm{ae} + 2\mathrm{bd} - 3\mathrm{c}^2$;

here
$$\mathrm{ae} = (B+K)^2 - D\Box - 4D\,(\Delta\beta'^2 - 2\Delta'\beta\beta' + \Delta''\beta^2),$$
$$2\mathrm{bd} = 2BK + 2B^2 - 2D\,(\Delta\beta'^2 - 2\Delta'\beta\beta' + \Delta''\beta^2),$$
$$-3\mathrm{c}^2 = -3B^2 - 2BK - \tfrac{1}{3}K^2.$$

Hence

$$\mathrm{ae} + 2\mathrm{bd} - 3\mathrm{c}^2 = \tfrac{2}{3}K^2 + 2BK - D\Box - 6D\,(\Delta\beta'^2 - 2\Delta'\beta\beta' + \Delta''\beta^2)$$
$$= -4D\,(\Delta\Delta'' - \Delta'^2 + \Delta\beta'^2 - 2\Delta'\beta\beta' + \Delta''\beta^2) + 2\mathrm{c}K$$
$$= D\{(\gamma\alpha' - \gamma'\alpha)^2 + 2(\alpha\beta' - \alpha'\beta)(\beta\gamma' - \beta'\gamma)\} + \tfrac{1}{3}6\mathrm{c}K.$$

The remaining terms be$-$cd, ce$-$d^2 are of course obtained by a mere interchange of letters; and, substituting the foregoing values, we find

$$\text{Quarticovariant} = D\begin{vmatrix} \eta^2, & -\eta\xi, & \xi^2 \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix}^2 + \tfrac{1}{3}KW$$
$$= D\Omega^2 + \tfrac{1}{3}KW,$$

where, as above, Ω is the Jacobian of V, V'.

The next derivative which we have to calculate is the cubinvariant,

$$\mathrm{ace} - \mathrm{ad}^2 - \mathrm{b}^2\mathrm{e} + 2\mathrm{bcd} - \mathrm{c}^3.$$

This, multiplied by 3, may also be written thus:

$$\mathrm{e}\,(\mathrm{ac} - \mathrm{b}^2) - 2\mathrm{d}\,(\mathrm{ad} - \mathrm{be}) + \mathrm{c}\,(\mathrm{ae} + 2\mathrm{bd} - 3\mathrm{c}^2) - 2\mathrm{b}\,(\mathrm{be} - \mathrm{cd}) + \mathrm{a}\,(\mathrm{ce} - \mathrm{d}^2),$$

and, substituting herein the values found above, this expression will become

$$\begin{aligned}
= \quad & \mathrm{e}\,(ac-b^2)\,(\alpha\beta'-\alpha'\beta)^2 && +\tfrac{1}{3}\mathrm{ae}K \\
& +2\mathrm{d}\,(ac-b^2)\,(\alpha\beta'-\alpha'\beta)\,(\gamma\alpha'-\gamma'\alpha) && -\tfrac{4}{3}\mathrm{bd}K \\
& +\ \mathrm{c}\,(ac-b^2)\,\{(\gamma\alpha'-\gamma'\alpha)^2+2\,(\alpha\beta'-\alpha'\beta)\,(\beta\gamma'-\beta'\gamma)\} && +2\mathrm{c}^2K \\
& +2\mathrm{b}\,(ac-b^2)\,(\beta\gamma'-\beta'\gamma)\,(\gamma\alpha'-\gamma'\alpha) && -\tfrac{4}{3}\mathrm{bd}K \\
& +\ \mathrm{a}\,(ac-b^2)\,(\beta\gamma'-\beta'\gamma)^2 && +\tfrac{1}{3}\mathrm{ae}K \\
= \ & D\,(\mathrm{a, c, e, d, c, b})\,(A, B, C)^2+\tfrac{2}{3}K\,(\tfrac{4}{3}K^2-D\square).
\end{aligned}$$

But $$(\mathrm{a, c, e, d, c, b})\,(A, B, C)^2$$

$$= (\mathrm{a}A+\mathrm{b}B+\mathrm{c}C)\,A+(\mathrm{b}A+\mathrm{c}B+\mathrm{d}C)\,B+(\mathrm{c}A+\mathrm{d}B+\mathrm{e}C)\,C,$$

and
$$\begin{aligned}
\mathrm{a}A+\mathrm{b}B+\mathrm{c}C = \ & (a, b, c)\,(\alpha, \alpha')^2 \quad A \\
& +(a, b, c)\,(\alpha, \alpha')\,(\beta, \beta')\,B \\
& +(a, b, c)\,(\alpha, \alpha')\,(\gamma, \gamma')\,C \\
& -\tfrac{2}{3}\,\{(a, b, c)(\alpha, \alpha')(\gamma, \gamma')-(a, b, c)(\beta, \beta')^2\}\,C \\
= & -\tfrac{2}{3}KC\,;
\end{aligned}$$

and, following the same process with the other terms, we should find that the whole expression

$$= -\tfrac{1}{3}K\,(4AC-B^2) = -\tfrac{1}{3}K\square.$$

Hence, finally, the cubinvariant in question

$$= \tfrac{1}{27}K\,(8K^2-3D\square). \qquad *$$

The cubicovariant may be written thus:

$$\begin{aligned}
& \{\mathrm{a}\,(\mathrm{ad}-\mathrm{be})-2\mathrm{b}\,(\mathrm{ac}-\mathrm{b}^2)\} && \xi^6 \\
+ & \{\mathrm{a}\,(\mathrm{ae}+2\mathrm{bd}-3\mathrm{c}^2)-6\mathrm{c}\,(\mathrm{ac}-\mathrm{b}^2)\} && \xi^5\eta \\
+ & \{\mathrm{a}\,(\mathrm{be}-\mathrm{cd})-2\mathrm{d}\,(\mathrm{ac}-\mathrm{b}^2)\} && \xi^4\eta^2 \\
+ & \{-\mathrm{d}\,(\mathrm{ad}-\mathrm{be})+\mathrm{b}\,(\mathrm{be}-\mathrm{cd})\} && \xi^3\eta^3 \\
+ & \{-\mathrm{e}\,(\mathrm{ad}-\mathrm{be})+2\mathrm{b}\,(\mathrm{ce}-\mathrm{d}^2\} && \xi^2\eta^4 \\
+ & \{-\mathrm{e}\,(\mathrm{ae}+2\mathrm{bd}-3\mathrm{c}^2)+6\mathrm{c}\,(\mathrm{ce}-\mathrm{d}^2)\} && \xi\,\eta^5 \\
+ & \{-\mathrm{e}\,(\mathrm{be}-\mathrm{cd})+2\mathrm{d}\,(\mathrm{ce}-\mathrm{d}^2)\} && \eta
\end{aligned}$$

Now, referring to former values;

$$\mathrm{a}\,(\mathrm{ad}-\mathrm{be})-2\mathrm{b}\,(\mathrm{ac}-\mathrm{b}^2)$$
$$=-\mathrm{a}DBC+\tfrac{2}{3}\mathrm{ab}K-2\mathrm{b}DC^2-\tfrac{2}{3}\mathrm{ab}K$$
$$=-DC\,(\mathrm{a}B+2\mathrm{b}C),$$

$$\mathrm{a}\,(\mathrm{ae}+2\mathrm{bd}-3\mathrm{c}^2)-6\mathrm{c}\,(\mathrm{ac}-\mathrm{b}^2)$$
$$=\mathrm{a}D\,(B^2+2AC)+2\mathrm{ac}K-6\mathrm{c}DC^2-2\mathrm{ac}K$$
$$=D\,\{-\mathrm{a}\,(4AC-B^2)+6C\,(\mathrm{a}A-\mathrm{c}C)\}$$
$$=6DC\,(\mathrm{a}A-\mathrm{c}C)-\mathrm{a}D\square,$$

$$\mathrm{a}\,(\mathrm{be}-\mathrm{cd})-2\mathrm{d}\,(\mathrm{ac}-\mathrm{b}^2)$$
$$=-\mathrm{a}DAB+\tfrac{2}{3}\mathrm{ad}K-2\mathrm{d}DC^2-\tfrac{2}{3}\mathrm{ad}K$$
$$=-D\,(\mathrm{a}AB+2\mathrm{d}C^2),$$

$$-\mathrm{d}\,(\mathrm{ad}-\mathrm{be})+\mathrm{b}\,(\mathrm{be}-\mathrm{cd})$$
$$=D\mathrm{d}BC-\tfrac{2}{3}\mathrm{bd}K-D\mathrm{b}AB+\tfrac{2}{3}\mathrm{bd}K$$
$$=DB\,(\mathrm{d}C-\mathrm{b}A),$$

$$-\mathrm{e}\,(\mathrm{ad}-\mathrm{be})+2\mathrm{b}\,(\mathrm{ce}-\mathrm{d}^2)$$
$$=D\mathrm{e}BC-\tfrac{2}{3}\mathrm{be}K+2D\,\mathrm{b}A^2+\tfrac{2}{3}\mathrm{be}K$$
$$=D\,(\mathrm{e}BC+2\mathrm{b}A^2),$$

$$-\mathrm{e}\,(\mathrm{ae}+2\mathrm{bd}-3\mathrm{c}^2)+6\mathrm{c}\,(\mathrm{ce}-\mathrm{d}^2)$$
$$=-D\mathrm{e}\,(B^2+2AC)-2\mathrm{ce}K+6\mathrm{c}DA^2+2\mathrm{ce}K$$
$$=-6DA\,(\mathrm{e}C-\mathrm{c}A)+\mathrm{e}D\square,$$

$$-\mathrm{e}\,(\mathrm{be}-\mathrm{cd})+2\mathrm{d}\,(\mathrm{ce}-\mathrm{d}^2)$$
$$=\mathrm{e}DAB-\tfrac{2}{3}\mathrm{ce}K+2\mathrm{d}DA^2+\tfrac{2}{3}\mathrm{ce}K$$
$$=DA\,(\mathrm{e}B+2\mathrm{d}A).$$

But it will further be found that

$$2\mathrm{b}C+\mathrm{a}B=(P,\,Q,\,R)\,(\alpha,\,\alpha')^2,$$
$$\mathrm{c}C-\mathrm{a}A=(P,\,Q,\,R)\,(\alpha,\,\alpha')\,(\beta,\,\beta')+\tfrac{1}{3}CK,$$
$$2\mathrm{d}C^2+\mathrm{a}AB=C\,(P,\,Q,\,R)\,(\beta,\,\beta')^2-B\,(P,\,Q,\,R)\,(\alpha,\,\alpha')\,(\beta,\,\beta'),$$
$$\mathrm{b}A-\mathrm{d}C=-(P,\,Q,\,R)\,(\beta,\,\beta')^2,$$
$$2\mathrm{b}A^2+\mathrm{e}BC=-A\,(P,\,Q,\,R)(\beta,\,\beta')^2+B\,(P,\,Q,\,R)\,(\beta,\,\beta')\,(\gamma,\,\gamma'),$$
$$\mathrm{c}A-\mathrm{e}C=-(P,\,Q,\,R)\,(\beta,\,\beta')\,(\gamma,\,\gamma')+\tfrac{1}{3}AK,$$
$$2\mathrm{d}A+\mathrm{e}B=-(P,\,Q,\,R)\,(\gamma,\,\gamma')^2.$$

Hence the cubicovariant $= -D\times$

$$
\begin{array}{ll}
C(P,Q,R)(a,a')^2 & \xi^6 \\
+\ 6C(P,Q,R)(a,a')(\beta,\beta')+2C^2K+\mathrm{a}\square & \xi^5\eta \\
+\ 5C(P,Q,R)(\beta,\beta')^2-5B(P,Q,R)(a,a')(\beta,\beta') & \xi^4\eta^2 \\
-10B(P,Q,R)(\beta,\beta')^2 & \xi^3\eta^3 \\
+\ 5A(P,Q,R)(\beta,\beta')^2\xi^2\eta^4-5B(P,Q,R)(\beta,\beta')(\gamma,\gamma') & \xi^2\eta^4 \\
+\ 6A(P,Q,R)(\beta,\beta')(\gamma,\gamma')-2A^2K-\mathrm{e}\square & \xi\eta^5 \\
+\ \ A(P,Q,R)(\gamma,\gamma')^2 & \eta^6.
\end{array}
$$

The expression contains terms divisible by $A\xi^2$, $B\xi\eta$, $C\eta^2$ respectively, and also the terms

$$(2C^2K+\mathrm{a}\square)\,\xi^5\eta-(2A^2K+\mathrm{e}\square)\,\xi\eta^5.$$

Now, since

$$(P,Q,R)(a,a')(\gamma,\gamma')-(P,Q,R)(\beta,\beta')^2=0,$$

it follows that the coefficient of $C\xi^2$ in the covariant may be written thus:

$$
\begin{aligned}
&\{(P,Q,R)(a,a')^2\xi^2+2(P,Q,R)(a,a')(\beta,\beta')\xi\eta \\
&\qquad\qquad +(P,Q,R)(a,a')(\gamma,\gamma')\eta^2\}\,\xi^2 \\
&+2\{(P,Q,R)(\beta,\beta')(a,a')\xi^2+2(P,Q,R)(\beta,\beta')\xi\eta \\
&\qquad\qquad +(P,Q,R)(\beta,\beta')(\gamma,\gamma')\eta^2\}\,\xi\eta \\
&+\ \{(P,Q,R)(\gamma,\gamma')(a,a')\xi^2+2(P,Q,R)(\gamma,\gamma')(\beta,\beta')\xi\eta \\
&\qquad\qquad +(P,Q,R)(\gamma,\gamma')^2\eta^2\}\,\eta^2 \\
&+2\ \ (P,Q,R)(a,a')(\beta,\beta')\xi^3\eta-(P,Q,R)(a,a')(\gamma,\gamma')\xi^2\eta^2 \\
&\qquad -4(P,Q,R)(\beta,\beta')(\gamma,\gamma')\xi\eta^3-(P,Q,R)(\gamma,\gamma')^2\eta^4 \\
&=(P,Q,R)(V,V')^2-(P,Q,R)(\gamma,\gamma')(V,V')\eta^2 \\
&\qquad +2(P,Q,R)(a,a')(\beta,\beta')\xi^3\eta-2(P,Q,R)(\beta,\beta')(\gamma,\gamma')\xi\eta^3.
\end{aligned}
$$

Similarly the coefficient of $B\xi\eta$ is

$$
\begin{aligned}
&=-(P,Q,R)(V,V')^2-(P,Q,R)(\beta,\beta')(V,V')\xi\eta \\
&+(P,Q,R)(a,a')^2\xi^4-2(P,Q,R)(a,a')(\gamma,\gamma')\xi^2\eta^2+(P,Q,R)(\gamma,\gamma')^2\eta^4,
\end{aligned}
$$

and that of $A\eta^2$ is

$$
\begin{aligned}
&=(P,Q,R)(V,V')^2-(P,Q,R)(a,a')(V,V')\xi^2 \\
&+2(P,Q,R)(\gamma,\gamma')(\beta,\beta')\xi\eta^3-2(P,Q,R)(a,a')(\beta,\beta')\xi^3\eta.
\end{aligned}
$$

Multiplying each of these results by their respective factors and adding the products together, the first terms give

$$\Omega(P,Q,R)(V,V')^2;$$

the second terms give

$$(P, Q, R)(V, V')(A\alpha+B\beta+C\gamma, A\alpha'+B\beta'+C\gamma') = 0;$$

the third terms give

$$(P, Q, R)(\beta, \beta')(A\alpha+B\beta+C\gamma, A\alpha'+B\beta'+C\gamma') = 0;$$

and there remain the terms

$$\{2C(P, Q, R)(\alpha, \alpha')(\beta, \beta')+B(P, Q, R)(\alpha, \alpha')^2+2C^2K+\mathrm{a}\square\}\xi^5\eta$$
$$+\{2A(P, Q, R)(\beta, \beta')(\gamma, \gamma')+B(P, Q, R)(\gamma, \gamma')^2-2A^2K-\mathrm{e}\square\}\xi\eta^5.$$

Now $2C(P, Q, R)(\alpha, \alpha')(\beta, \beta')+B(P, Q, R)(\alpha, \alpha')^2+\mathrm{a}\square$

$$= (P, Q, R)(\alpha, \alpha')(2C\beta+B\alpha, 2C\beta'+B\alpha')+\mathrm{a}\square$$
$$= 2(P, Q, R)(\alpha, \alpha')(\alpha'\Delta-\alpha\Delta', \alpha'\Delta'-\alpha\Delta'')+\mathrm{a}\square$$
$$\begin{aligned} = 2 \times c \,.\, \Delta \,.\, \alpha\alpha'\Delta - \alpha^2\Delta' &+ 4\mathrm{a}(\Delta\Delta''-\Delta'^2) \\ -2b \,.\, 2\Delta' \,.\, \alpha'^2\Delta - \alpha^2\Delta'' & \\ + a \,.\, \Delta'' \,.\, \alpha'^2\Delta'-\alpha\alpha'\Delta'', & \end{aligned}$$

which, on being developed, will be found to be $= -2KC^2$. Hence the terms in $\xi^5\eta$ and $\xi\eta^5$ will vanish, and the cubicovariant in question finally found to be $\quad = -D\Omega(P, Q, R)(V, V')^2$.

Recapitulating, we have, for the derivatives of W

the Quadrinvariant, $\quad I = \frac{4}{3}K^2 - D\square$,

the Quarticovariant, $\quad H = D\Omega+\frac{1}{3}KW$,

the Cubinvariant, $\quad J = \frac{1}{27}K(8K^2-3D\square)$,

the Cubicovariant, $\quad \Phi = -D\Omega(P, Q, R)(V, V')^2$;

whence also for the Discriminant, we have

$$I^3-27J^2 = \tfrac{1}{3}D\square(-32K^4+33K^2D\square-9D^2\square^2).$$

If $(P, Q, R)(V, V')^2$ be regarded as a quartic in ξ, η, then its

Quadrinvariant $= \frac{4}{3}(P\Delta+2Q\Delta'+R\Delta'') - \square(PR-Q^2)$,

which, since $P\Delta+2Q\Delta'+R\Delta'' = 0$, is

$$= -\square(D\square-K^2);$$

Quarticovariant $= \quad \Omega(D\square-K^2)$,

Cubinvariant $\quad = \quad 0$;

Cubicovariant $\quad = -\Omega(D\square-K^2)(P_1, Q_1, R_1)(V, V')^2$,

where P_1, Q_1, R_1 are quantities formed with P, Q, R, in the same way as were P, Q, R with a, b, c; viz.,

$$P_1,\ Q_1,\ R_1 = \left\| \begin{matrix} R, & -2Q, & P \\ \Delta, & 2\Delta', & \Delta'' \end{matrix} \right\|;$$

and, on developing these expressions, it will be found that

$$\begin{aligned} P_1 &= 2\{-\ a(\Delta\Delta''-\Delta'^2)+(a\Delta'^2+2b\Delta'\Delta''+c\Delta''^2)\}, \\ Q_1 &= 2\{-2b(\Delta\Delta''-\Delta'^2)-(a\Delta\Delta'+2b\Delta'^2+c\Delta'\Delta'')\}, \\ R_1 &= 2\{-\ c(\Delta\Delta''-\Delta'^2)+(a\Delta^2+2b\Delta\Delta'+c\Delta'^2)\}; \end{aligned}$$

but

$$\begin{aligned} a\Delta'^2+2b\Delta'\Delta''+c\Delta''^2 &= K\Delta''-\tfrac{1}{4}a\Box, \\ a\Delta\Delta'+2b\Delta'^2+c\Delta'\Delta'' &= K\Delta', \\ a\Delta^2+2b\Delta\Delta'+c\Delta'^2 &= K\Delta-\tfrac{1}{4}c\Box; \end{aligned}$$

hence

$$\begin{aligned} P_1 &= 2K\Delta''-a\Box, \\ Q_1 &= -2K\Delta'-b\Box, \\ R_1 &= 2K\Delta-c\Box. \end{aligned}$$

But, on referring to the expression for W given on p. 155, and replacing a, b, c by Δ'', $-\Delta'$, Δ, respectively, it will be found that the

coefficient of

$$\begin{aligned} \xi^4 &= -\ C^2, \\ \xi^3\eta &= 2BC, \\ \xi^2\eta^2 &= -(2AC+B^2), \\ \xi\eta^3 &= 2AB, \\ \eta^4 &= -\ A^2, \end{aligned}$$

so that the whole coefficient of K is $=-(C\xi^2-B\xi\eta+A\eta^2)^2$. Hence

$$(P_1,\ Q_1,\ R_1)(V,\ V')^2 = -K\Omega-2\Box\,(P,\ Q,\ R)(V,\ V')^2,$$

and the cubicovariant in question

$$=\Omega\,(D\Box-K^2)\,\{K\Omega+2\Box\,(P,\ Q,\ R)(V,\ V')^2\}.$$

If the quartic W has a pair of equal roots, then

$I^3-27J^2=0$, and either $\Box=0$, or $(32,\ -33,\ 9)(K^2,\ D\Box)^2=0$.

The condition $\Box=0$ must, however, be excluded, since it implies

that V, V' have a common factor; in which case the transformation ceases to be quadratic. For this factor may be divided out from the transformant, and the transformation then becomes linear.

If it has two pairs of equal roots, then

$$\Phi = 0, \text{ and either } \Omega = 0, \text{ or } (P, Q, R)(V, V')^2 = 0.$$

If it has three equal roots,

$$I = 0, \text{ and } J = 0, \text{ and then } K = 0, \ \square = 0,$$

viz., we have here the excluded condition.

If all the roots are equal, then

$$H = 0, \text{ and } \Omega = 0, \ K = 0.$$

As regards the quartic $(P, Q, R)(V, V')^2$, if $D\square - K^2 = 0$, *i.e.*, if the two quadrics U, U' have a common factor, then all the invariants and covariants vanish, and all the roots of the quartic in question become equal; viz.,

$$(P, Q, R)(V, V')^2 = (V\sqrt{P} + V'\sqrt{R})^2$$
$$= \{(\alpha\sqrt{P} + \alpha'\sqrt{R}, \beta\sqrt{P} + \beta'\sqrt{R}, \gamma\sqrt{P} + \gamma'\sqrt{R})(\xi, \eta)^2\}^2.$$

But the discriminant of the last form

$$= P\Delta + 2Q\Delta' + R\Delta'' = 0,$$

or we have $V\sqrt{P} + V'\sqrt{R}$ a perfect square, and hence the quantic

$$= \{\sqrt{(\alpha\sqrt{P} + \alpha'\sqrt{R})}\,\xi + \sqrt{(\gamma\sqrt{P} + \gamma'\sqrt{R})}\,\eta\}^4.$$

We now proceed to the cubic

$$U = (a, b, c, d)(x, y)^3,$$

which, if transformed by the same formula as was the quadratic, gives

$$W = (a, b, c, d)(\alpha\xi^2 + 2\beta\xi\eta + \gamma\eta^2, \alpha'\xi^2 + 2\beta'\xi\eta + \gamma'\eta^2)^3$$

and if we represent this by

$$(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{f}, \mathrm{g})(\xi, \eta)^6,$$

we shall have

$$\mathrm{a} = (a,\ldots)(a,\,a')^3,$$
$$\mathrm{b} = (a,\ldots)(a,\,a')^2(\beta,\,\beta'),$$
$$\mathrm{c} = (a,\ldots)(a,a')(\beta,\beta')^2 + \tfrac{1}{3}\{(a,\ldots)(a,a')^2(\gamma,\gamma') - (a,\ldots)(a,a')(\beta,\beta')^2\},$$
$$\mathrm{d} = (a,\ldots)(\beta,\,\beta')^3 + \tfrac{2}{3}\{(a,\,\ldots)(a,\,a')(\beta,\,\beta')(\gamma,\,\gamma') - (a,\,\ldots)(\beta,\,\beta')^3\},$$
$$\mathrm{e} = (a,\ldots)(\gamma,\gamma')(\beta,\beta')^2 + \tfrac{1}{3}\{(a,\ldots)(a,a')(\gamma,\gamma')^2 - (a,\ldots)(\beta,\beta')^2(\gamma,\gamma')\},$$
$$\mathrm{f} = (a,\ldots)(\gamma,\,\gamma')^2(\beta,\,\beta'),$$
$$\mathrm{g} = (a,\ldots)(\gamma,\,\gamma')^3.$$

Further, if we put

$$\mathrm{h} = (a,\,b,\,c,\,d)(\beta,\,\beta')^3,$$
$$\mathrm{h}_1 = (a,\,b,\,c)(\beta,\,\beta')^2,\quad \mathrm{h}_1' = (b,\,c,\,d)(\beta,\,\beta')^2,$$
$$\mathrm{h}_2 = (a,\,b)(\beta,\,\beta'),\quad \mathrm{h}_2' = (b,\,c)(\beta,\,\beta'),\quad \mathrm{h}_2'' = (c,\,d)(\beta,\,\beta'),$$
$$a\Delta + 2b\Delta' + c\Delta'' = K_1,$$
$$b\Delta + 2c\Delta' + d\Delta'' = K_1',$$
$$(ac-b^2)\,\Delta + (ad-bc)\,\Delta' + (bd-c^2)\,\Delta'' = K = L\Delta + M\Delta' + N\Delta'';$$

then

$$\mathrm{a} = (a,\,b,\,c,\,d)(a,\,a')^3,$$
$$\mathrm{b} = (\mathrm{h}_2,\,\mathrm{h}_2',\,\mathrm{h}_2'')(a,\,a')^2,$$
$$\mathrm{c} = (\mathrm{h}_1 + \tfrac{1}{3}K_1,\,\mathrm{h}_1' + \tfrac{1}{3}K_1')(a,\,a'),$$
$$\mathrm{d} = \mathrm{h} + \tfrac{2}{3}(K_1\beta + K_1'\beta'),$$
$$\mathrm{e} = (\mathrm{h}_1 + \tfrac{1}{3}K_1,\,\mathrm{h}_1' + \tfrac{1}{3}K_1')(\gamma,\,\gamma'),$$
$$\mathrm{f} = (\mathrm{h}_2,\,\mathrm{h}_2',\,\mathrm{h}_2'')(\gamma,\,\gamma')^2,$$
$$\mathrm{g} = (a,\,b,\,c,\,d)(\gamma,\,\gamma')^3.$$

The quadrinvariant

$$\mathrm{ag} - 6\mathrm{bf} + 15\mathrm{ce} - 10\mathrm{d}^2,$$

of this function is found to be

$$= \tfrac{8}{3}\{\Delta K_1'^2 + 2\Delta' K_1 K_1' + \Delta'' K_1^2 - 5K(\Delta\Delta'' - \Delta'^2)\}.$$

Given the Quantic

$$U = (a,\,b,\,c,\,f,\,g,\,h)\ (x,\,y,\,z)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1$$

and $$V = (a, \beta, \gamma, \lambda, \mu, \nu)(\xi, \eta, \zeta)^2,$$

and like values of V', V'', or, more briefly, say

$$V = (a, \ldots)(\xi, \eta, \zeta)^2,\ V' = (a', \ldots)(\xi, \eta, \zeta)^2,\ V'' = (a'', \ldots)(\xi, \eta, \zeta)^2 \ldots (2),$$

then U may be transformed by the equations

$$x : y : z = V : V' : V'' \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

The result is $$W = (a, \ldots)(V, V', V'')^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

and, if we write for brevity

$$\left.\begin{array}{c} (a)^2 = (a, \ldots)(a, a', a'')^2 \\ (a)(\beta) = (a, \ldots)(a, a', a'')(\beta, \beta', \beta'') \\ \cdots \quad \cdots \quad \cdots \end{array}\right\} \ldots\ldots\ldots\ldots (5),$$

then $$\left.\begin{array}{l} W = (a)^2\xi^4 + (\beta)^2\eta^4 + (\gamma)^2\zeta^4 \\ \quad +4(\beta)(\lambda)\ \eta^3\zeta + 4(\gamma)(\lambda)\ \eta\zeta^3 \\ \quad +4(\gamma)(\mu)\ \zeta^3\xi + 4(a)(\mu)\ \zeta\xi^3 \\ \quad +4(a)(\nu)\ \xi^3\eta + 4(\beta)(\nu)\ \xi\eta^3 \\ \quad +2\left[(\beta)(\gamma) + 2(\lambda)^2\right]\eta^2\zeta^2 \\ \quad +2\left[(\gamma)(a) + 2(\mu)^2\right]\zeta^2\xi^2 \\ \quad +2\left[(a)(\beta) + 2(\nu)^2\right]\xi^2\eta^2 \\ \quad +4\left[(a)(\lambda) + 2(\mu)(\nu)\right]\xi^2\eta\zeta \\ \quad +4\left[(\beta)(\mu) + 2(\nu)(\lambda)\right]\xi\eta^2\zeta \\ \quad +4\left[(\gamma)(\nu) + 2(\lambda)(\mu)\right]\xi\eta\zeta^2 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

It is proposed to express the invariants, &c. of W in terms of those of U, V, V', V''. If we write the above value of W in the following form $W = (\mathrm{a, b, c, f, g, h, p, p', q, q', r, r', l, m, n})(\xi, \eta, \zeta)^4$,

we shall have $$\mathrm{a} = (a)^2,\quad \mathrm{b} = (\beta)^2,\quad \mathrm{c} = (\gamma)^2,$$

$$3\mathrm{f} = (\beta)(\gamma) + 2(\lambda)^2,$$

$$3\mathrm{g} = (\gamma)(a) + 2(\mu)^2,$$

$$3\mathrm{h} = (a)(\beta) + 2(\nu)^2,$$

$$\mathrm{p} = (\beta)(\lambda),\qquad \mathrm{p}' = (\gamma)(\lambda),$$

$$\mathrm{q} = (\gamma)(\mu),\qquad \mathrm{q}' = (a)(\mu),$$

$$\mathrm{r} = (a)(\nu),\qquad \mathrm{r}' = (\beta)(\nu),$$

$$3l = (\alpha)(\lambda) + 2(\mu)(\nu),$$
$$3m = (\beta)(\mu) + 2(\nu)(\lambda),$$
$$3n = (\gamma)(\nu) + 2(\lambda)(\mu),$$

and the quantities which we have first to calculate are

$$\mathbb{A} = bc + 3f^2 - 4pp',$$
$$\mathbb{B} = ca + 3g^2 - 4qq',$$
$$\mathbb{C} = ab + 3h^2 - 4rr',$$
$$\mathbb{F} = af + gh + 2l^2 - 2rn - 2q'm,$$
$$\mathbb{G} = bg + hf + 2m^2 - 2pl - 2r'n,$$
$$\mathbb{H} = ch + fg + 2n^2 - 2qm - 2p'l,$$
$$\mathbb{L} = 2fl - mn - gp - hp' + qr',$$
$$\mathbb{M} = 2gm - nl - hq - fq' + rp',$$
$$\mathbb{N} = 2hn - lm - fr - gr' + pq',$$

$$\mathbb{P} = 3nq' - 3lg - ap' + qr, \qquad \mathbb{P}' = 3mr - 3lh - ap + q'r',$$
$$\mathbb{Q} = 3lr' - 3mh - bq' + rp, \qquad \mathbb{Q}' = 3np - 3mf - bq + r'p',$$
$$\mathbb{R} = 3mp' - 3nf - cr' + pq, \qquad \mathbb{R}' = 3lq - 3ng - cr + p'q'.$$

[*Mem.*—Comparing the notation of Salmon's *Higher Plane Curves* with that used here, we have

$$p = b_3, \quad p' = c_2, \qquad P = B_3, \quad P' = C_2,$$
$$q = c_1, \quad q' = a_3, \qquad Q = C_1, \quad Q' = A_3,$$
$$r = a_2, \quad r' = b_1, \qquad R = A_2, \quad R' = B_1.]$$

If we now put

$$\beta\gamma - \lambda^2 = A, \quad \gamma\alpha - \mu^2 = B, \quad \alpha\beta - \nu^2 = C,$$
$$\mu\nu - \alpha\lambda = F, \quad \nu\lambda - \beta\mu = G, \quad \lambda\mu - \gamma\nu = H,$$

and
$$\beta'\gamma' - \lambda'^2 = A', \ldots \quad \beta'\gamma'' + \beta''\gamma' - 2\lambda'\lambda'' = A^{\backslash}, \ldots$$
$$aA + bA' + cA'' + 2fA^{\backslash} + 2gA^{\backslash\backslash} + 2hA^{\backslash\backslash\backslash} = \mathbf{A}, \ldots$$
$$\vdots$$
$$aF + bF' + cF'' + 2fF^{\backslash} + 2gF^{\backslash\backslash} + 2hF^{\backslash\backslash\backslash} = \mathbf{F},$$
$$\vdots$$
$$bc - f^2 = \mathfrak{A}, \quad ca - g^2 = \mathfrak{B}, \quad ab - h^2 = \mathfrak{C},$$
$$gh - af = \mathfrak{F}, \quad hf - bg = \mathfrak{G}, \quad fg - ch = \mathfrak{H},$$

then $\frac{3}{4}\cdot\mathbb{A} = \mathbf{A}^2 - 3\mathfrak{A}(A'A'' - A^{\backprime 2}) - 6\mathfrak{F}(A^{\backprime\backprime}A^{\backprime\backprime\backprime} - AA^{\backprime}),\ \frac{3}{4}\mathbb{B} = \ldots,\ \frac{3}{4}\mathbb{C} = \ldots,$

$$-3\mathfrak{B}(A''A - A^{\backprime\backprime 2}) - 6\mathfrak{G}(A^{\backprime\backprime\backprime}A^{\backprime} - A'A^{\backprime\backprime})$$

$$-3\mathfrak{C}(AA' - A^{\backprime\backprime\backprime 2}) - 6\mathfrak{H}(A^{\backprime}A^{\backprime\backprime} - A''A^{\backprime\backprime\backprime}),$$

$$\frac{3}{4}\cdot 6\mathbb{F} = \mathbf{BC} + 4\mathbf{F}^2 \qquad ,\ \frac{3}{4}\cdot 6\mathbb{G} = \ldots,\ \frac{3}{4}\cdot 6\mathbb{H} = \ldots,$$

$$-3\mathfrak{A}(B'C'' + B''C' - 2B^{\backprime}C^{\backprime} + 4F'F'' - 4F^{\backprime 2})$$

$$-3\mathfrak{B}(B''C + BC'' - 2B^{\backprime\backprime}C^{\backprime\backprime} + 4F''F - 4F^{\backprime\backprime 2})$$

$$-3\mathfrak{C}(BC' + B'C - 2B^{\backprime\backprime\backprime}C^{\backprime\backprime\backprime} + 4FF' - 4F^{\backprime\backprime\backprime 2})$$

$$-6\mathfrak{F}(B^{\backprime\backprime}C^{\backprime\backprime\backprime} + B^{\backprime\backprime\backprime}C^{\backprime} - BC^{\backprime} - B^{\backprime}C + 4F^{\backprime\backprime}F^{\backprime\backprime\backprime} - 4FF^{\backprime})$$

$$-6\mathfrak{G}(B^{\backprime\backprime\backprime}C^{\backprime} + B^{\backprime}C^{\backprime\backprime\backprime} - B'C^{\backprime\backprime} - B^{\backprime\backprime}C' + 4F^{\backprime\backprime\backprime}F^{\backprime} - 4F'F^{\backprime\backprime})$$

$$-6\mathfrak{H}(B^{\backprime}C^{\backprime\backprime} + B^{\backprime\backprime}C^{\backprime} - B'C^{\backprime\backprime\backprime} - B^{\backprime\backprime\backprime}C'' + 4F^{\backprime}F^{\backprime\backprime} - 4F''F^{\backprime\backprime\backprime}),$$

$$\frac{3}{4}\cdot 12\mathbb{L} = \mathbf{AF} + 8\mathbf{GH} \qquad ,\ \frac{3}{4}\cdot 12\mathbb{M} = \ldots,\ \frac{3}{4}\cdot 12\mathbb{N} = \ldots$$

$$-\ 6\mathfrak{A}(A'F'' + A''F' - 2A^{\backprime}F^{\backprime} + 2G'H'' + 2G''H' - 4G^{\backprime}H^{\backprime})$$

$$-\ 6\mathfrak{B}(A''F + AF'' - 2A^{\backprime\backprime}F^{\backprime\backprime} + 2G''H + 2GH'' - 4G^{\backprime\backprime}H^{\backprime\backprime})$$

$$-\ 6\mathfrak{C}(AF' + A'F - 2A^{\backprime\backprime\backprime}F^{\backprime\backprime\backprime} + 2GH' + 2G'H - 4G^{\backprime\backprime\backprime}H^{\backprime\backprime\backprime}),$$

$$-12\mathfrak{F}(A^{\backprime\backprime}F^{\backprime\backprime\backprime} + A^{\backprime\backprime\backprime}F^{\backprime\backprime} - AF^{\backprime} - A^{\backprime}F + 2G^{\backprime\backprime}H^{\backprime\backprime\backprime} + 2G^{\backprime\backprime\backprime}H^{\backprime\backprime} - 2GH^{\backprime} - 2G^{\backprime}H)$$

$$-12\mathfrak{G}(A^{\backprime\backprime\backprime}F^{\backprime} + A^{\backprime}F^{\backprime\backprime\backprime} - A'F^{\backprime\backprime} - A^{\backprime\backprime}F' + 2G^{\backprime\backprime\backprime}H^{\backprime} + 2G^{\backprime}H^{\backprime\backprime\backprime} - 2G'H^{\backprime\backprime} - 2G^{\backprime\backprime}H')$$

$$-12\mathfrak{H}(A^{\backprime}F^{\backprime\backprime} + A^{\backprime\backprime}F^{\backprime} - A''F^{\backprime\backprime\backprime} - A^{\backprime\backprime\backprime}F'' + 2G^{\backprime}H^{\backprime\backprime} + 2G^{\backprime\backprime}H^{\backprime} - 2G''H^{\backprime\backprime\backprime} - 2G^{\backprime\backprime\backprime}H''),$$

$$\frac{3}{4}\cdot 4\mathbb{P} = 4\mathbf{BF} - 6\mathfrak{A}(B'F'' + B''F' - 2B^{\backprime}F^{\backprime}),\ \frac{3}{4}\cdot 4\mathbb{Q} = \ldots,\ \frac{3}{4}\cdot 4\mathbb{R} = \ldots$$

$$-\ 6\mathfrak{B}(B''F + BF'' - 2B^{\backprime\backprime}F^{\backprime\backprime})$$

$$-\ 6\mathfrak{C}(BF' + B'F - 2B^{\backprime\backprime\backprime}F^{\backprime\backprime\backprime})$$

$$-12\mathfrak{F}(B^{\backprime\backprime}F^{\backprime\backprime\backprime} + B^{\backprime\backprime\backprime}F^{\backprime\backprime} - BF^{\backprime} - B^{\backprime}F)$$

$$-12\mathfrak{G}(B^{\backprime\backprime\backprime}F^{\backprime} + B^{\backprime}F^{\backprime\backprime\backprime} - B'F^{\backprime\backprime} - B^{\backprime\backprime}F')$$

$$-12\mathfrak{H}(B^{\backprime}F^{\backprime\backprime} + B^{\backprime\backprime}F^{\backprime} - B''F^{\backprime\backprime\backprime} - B^{\backprime\backprime\backprime}F'')$$

$$\frac{3}{4}\cdot 4\mathbb{P}' = 4\mathbf{CF} - 6\mathfrak{A}(C'F'' + C''F' - 2C^{\backprime}F^{\backprime}), \qquad \frac{3}{4}\cdot 4\mathbb{Q}' = \ldots,\ \frac{3}{4}\cdot 4\mathbb{R}'',$$

$$-\ 6\mathfrak{B}(C''F + CF'' - 2C^{\backprime\backprime}F^{\backprime\backprime})$$

$$-\ 6\mathfrak{C}(CF' + C'F - 2C^{\backprime\backprime\backprime}F^{\backprime\backprime\backprime})$$

$$-12\mathfrak{F}(C^{\backprime\backprime}F^{\backprime\backprime\backprime} + C^{\backprime\backprime\backprime}F^{\backprime\backprime} - CF^{\backprime} - C^{\backprime}F)$$

$$-12\mathfrak{G}(C^{\backprime\backprime\backprime}F^{\backprime} + C^{\backprime}F^{\backprime\backprime\backprime} - C'F^{\backprime\backprime} - C^{\backprime\backprime}F')$$

$$-12\mathfrak{H}(C^{\backprime}F^{\backprime\backprime} + C^{\backprime\backprime}F^{\backprime} - C''F^{\backprime\backprime\backprime} - C^{\backprime\backprime\backprime}F'').$$

Again, the invariant $(a\mathbb{A}+b\mathbb{B}+\ldots)$

$$= \tfrac{3}{4}(a, b, c, f, g, h, f, g, h, l, m, n, l, p, p', m\, q, q', n, r, r')(\mathbf{A}, \mathbf{B}, \mathbf{C}, 2\mathbf{F}, 2\mathbf{G}, 2\mathbf{H})^2$$
$$-3\mathfrak{A}(a, b, \ldots)(A', B', C', 2F', 2G', 2H')(A'', B''\ldots) \quad -\ldots$$
$$+3\mathfrak{A}(a, b, \ldots)(A', B', C', 2F', 2G', 2H')^2 \quad +\ldots$$
$$-6\mathfrak{F}(a, b, \ldots)(A''', B''', \ldots)(A'''', B'''', \ldots) \quad -\ldots$$
$$+6\mathfrak{F}(a, b, \ldots)(A, B, \ldots)(A', B', \ldots) \quad +\ldots$$

To find whether a quadratic transformation be possible having the following property.

Given $x : y = (a, \ldots)(\xi, \eta)^2 : (a', \ldots)(\xi, \eta)^2 = Y : Y'$,

and $\xi : \eta = (a, \ldots)(x, y)^2 : (a', \ldots)(x, y)^2 = U : U'$.

Then $(ay-a'x,\ \beta y-\beta' x,\ \gamma y-\gamma' x)(U, U')^2 = 0$,

$$\begin{aligned} i.e.,\ 0 = &\ (ay-a'x)\{a^2x^4+4abx^3y+2(ac+2b^2)x^2y^2+4bcxy^3+c^2y^4\} \\ &+2(\beta y-\beta' x)\{aa'x^4+2(ab'+a'b)x^3y+(ac'+a'c+4bb')x^2y^2+2(bc'+b'c)xy^3+cc'y^4\} \\ &+(\gamma y-\gamma' x)\{a'^2x^4+4a'b'x^3y+2(a'c'+2b'^2)x^2y^2+4b'c'xy^3+c'^2y^4\} \end{aligned}$$

$i.e.,\ 0 =$

$$\begin{aligned} &\{-(a',\beta',\gamma')(a,a')^2\}x^5 \\ &\{-4(a',\beta',\gamma')(a,a')(b,b') + (a,\beta,\gamma)(a,a')^2\}x^4y \\ &\{-2(a',\beta',\gamma')(a,a')(c,c')-4(a',\beta',\gamma')(b,b')^2+4(a,\beta,\gamma)(a,a')(b,b')\}x^3y^2 \\ &\{-4(a',\beta',\gamma')(b,b')(c,c') + 2(a,\beta,\gamma)(a,a')(c,c')+4(a,\beta,\gamma)(b,b')^2\}x^2y^3 \\ &\{-(a',\beta',\gamma')(c,c')^2 + 4(a,\beta,\gamma)(b,b')(c,c')\}xy^4 \\ &\{ + (a,\beta,\gamma)(c,c')^2\}y^5 \end{aligned}$$

If this is to hold good for all values of $x : y$, we must have

$$\begin{aligned} 0 &= (a',\beta',\gamma')(a,a')^2, \\ 0 &= 4(a',\beta',\gamma')(a,a')(b,b') - (a,\beta,\gamma)(a,a')^2, \\ 0 &= 2(a',\beta',\gamma')(a,a')(c,c')+4(a',\beta',\gamma')(b,b')^2-4(a,\beta,\gamma)(a,a')(b,b'), \\ 0 &= 4(a',\beta',\gamma')(b,b')(c,c') -2(a,\beta,\gamma)(a,a')(c,c')-4(a,\beta,\gamma)(b,b') \\ 0 &= (a',\beta',\gamma')(c,c')^2 -4(a,\beta,\gamma)(b,b')(c,c'), \\ 0 &= (a,\beta,\gamma)(c,c')^2. \end{aligned}$$

Whence, eliminating α', β', γ', α, β, γ, we have $0 =$

$$\begin{vmatrix} a^2, & 2aa', & a'^2, & . & . & . \\ 4ab, & 4(ab'+a'b), & 4a'b', & a^2, & 2aa', & a'^2 \\ ac+2b^2, & ac'+a'c+4bb', & a'c'+2b'^2, & 2ab, & 2(ab'+a'b), & 2a'b' \\ 2bc, & 2(bc'+b'c), & 2b'c', & ac+2b^2, & ac'+a'c+4bb', & a'c'+2b'^2 \\ c^2, & 2cc', & c'^2, & 4bc, & 4(bc'+b'c), & 4b'c' \\ . & . & . & c^2, & 2cc', & c'^2 \end{vmatrix}$$

Writing $2a'$ (col. 1)$-a$ (col. 2) for col. 1, we obtain

$$2a'(1)-a(2) = \begin{array}{c} 0 \\ -4aC \\ -4bC+aB \\ -4cC-2aA \\ 2cB \\ 0 \end{array} \qquad 2a(3)-a'(2) = \begin{array}{c} 0 \\ 4a'C \\ 4b'C-a'B \\ 4c'C+2a'A \\ -2c'B \\ 0. \end{array}$$

Similarly,

$$2c'(4)-c(5) = \begin{array}{c} 0 \\ -2aB \\ 2cC+4aA \\ -cB+4bA \\ 4cA \\ 0 \end{array} \qquad 2c(6)-c'(5) = \begin{array}{c} 0 \\ 2a'B \\ -2c'C-4a'A \\ c'B-4b'A \\ -4c'A \\ 0. \end{array}$$

Hence the whole determinant is

$$= \begin{vmatrix} +4aC, & 4a'C, & -2aB, & -2a'B \\ +4bC-aB, & 4b'C-a'B, & 2cC+4aA, & +2c'C+4a'A \\ +4cC+2aA, & 4c'C+2a'A, & -cB+4bA, & -c'B+4b'A \\ -2cB, & -2c'B, & 4cA, & +4c'A \end{vmatrix}.$$

Now taking columns 1 and 2, then the union formed from lines

$$1, 2 = \begin{matrix} 4aC & 4a'C \\ 4bC-aB & 4b'C-a'B \end{matrix} = 16C^3,$$

$$1, 3 = \begin{matrix} 4aC & 4a'C \\ 4cC+2aA & 4c'C+2a'A \end{matrix} = -16BC^2,$$

$$1,4 = \begin{matrix} 4aC & 4a'C \\ -2cB & -2c'B \end{matrix} = 8B^2C,$$

$$2,3 = \begin{matrix} 4bC-aB & 4b'C-a'B \\ 4cC+2aA & 4c'C+2a'A \end{matrix} = \begin{matrix} 16AC^2+8a'bAC-4c'aBC \\ -8ab'AC+4ca'BC \end{matrix}$$
$$= 8AC^2+4B^2C,$$

$$2,4 = \begin{matrix} 4bC-aB & 4b'C-a'B \\ -2cB & -2c'B \end{matrix} = -8ABC-2B^3,$$

$$3,4 = \begin{matrix} 4cC+2aA & 4c'C+2a'A \\ -2cB & -2c'B. \end{matrix} = 4AB^2,$$

Similarly, taking the last two columns,

$$1,2' = \begin{matrix} -2aB & -2a'B \\ 2cC+4aA & 2c'C+4a'A \end{matrix} = 4B^2C,$$

$$1,3' = \begin{matrix} -2aB & -2a'B \\ -cB+4bA & -cB+4b'A \end{matrix} = -2B^3-8ABC,$$

$$1,4' = \begin{matrix} -2aB & -2a'B \\ 4cA & 4c'A \end{matrix} = 8AB^2,$$

$$2,3' = \begin{matrix} 2cC+4aA & 2c'C+4a'A \\ -cB+4bA & -c'B+4b'A \end{matrix} = 8A^2C+4AB^2,$$

$$2,4' = \begin{matrix} 2cC+4aA & 2c'C+4a'A \\ 4cA & 4c'A \end{matrix} = -16A^2B,$$

$$3,4' = \begin{matrix} -cB+4bA & -c'B+4b'A \\ 4cA & 4c'A. \end{matrix} = 16A^3,$$

Hence, for the whole determinant, we have

$$\begin{aligned} 12.34' &= 16^2A^3C^3, \\ -13.24' &= -16^2A^3B^2C^2, \\ +14.23' &= 32AB^2C\,(B^2+2AC), \\ +23.14' &= 32AB^2C\,(B^2+2AC), \\ -24.13' &= -4B^2\,(B^2+4AC)^2, \\ +34.12' &= 16AB^4C. \end{aligned}$$

The sum of which, rejecting the common factor 4, is

$$\begin{aligned}
&= 4^3A^3C^3-4\,.\,4^2A^2C^2\,.\,B^2+2\,.\,4AC\,.\,B^4\\
&\qquad +\ 4^2A^2C^2\,.\,B^2+2\,.\,4AC\,.\,B^4\\
&\qquad +\ 4^2A^2C^2\,.\,B^2\\
&\qquad -\ 4^2A^2C^2\,.\,B^2-2\,.\,4AC\,.\,B^4-B^6\\
&\qquad\qquad\qquad +\ 4AC\,.\,B^4,\\
&= 4^3A^3C^3-3\,.\,4^2A^2C^2\,.\,B^2+3\,.\,4AC\,.\,B^4-B^6\\
&= (4AC-B^2)^3.
\end{aligned}$$

Hence the condition sought is $4AC-B^2=0$; *i.e.*, the two quadrics U, U' must have a common factor. Similarly, we should find that Υ, Υ' must have a common factor. In other words, throwing out this common factor in each case, we have

$$\xi : \eta = uv : uv',$$

$$x : y = wv : wv',$$

$$x : y = \lambda\xi+\mu\eta : \lambda'\xi+\mu'\eta,$$

$$\xi : \eta = lx+my : l'x+m'y.$$

The Differential Equations of Cylindrical and Annular Vortices.

By M. J. M. Hill, M.A., Professor of Mathematics at University College, London.

[*Read January* 8*th*, 1885.]

Abstract.

I. *Cylindrical Vortices.*

The current function Λ of the motion of a fluid in two dimensions being known to satisfy the equation

$$\left(\frac{d}{dt}+\frac{d\Lambda}{dy}\frac{d}{dx}-\frac{d\Lambda}{dx}\frac{d}{dy}\right)\left(\frac{d^2\Lambda}{dx^2}+\frac{d^2\Lambda}{dy^2}\right)=0,$$

it is shown in this paper how to obtain the equation satisfied by ζ, the molecular rotation.

It is exhibited in the form

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\int \delta x \left\{\frac{\frac{d\zeta}{dt}}{\frac{d\zeta}{dy}}\right\}_{t}^{\substack{x\\ \zeta}} = \zeta$$

(provided that ζ be not constant throughout the fluid), where in the integral the expression in brackets is expressed as a function of x, ζ, t by eliminating y by means of the expression for ζ in terms of x, y, t supposed known; and the integration is performed with regard to x, treating ζ and t as constant, so that the integral is arbitrary to the extent of a function of ζ and t.

Exhibited as an ordinary partial differential equation, this would be of the fourth order, but the form is interesting, because the integral represents the space passed over by an arc of the line $\zeta=$ const., as might be expected from the fact that the integral is the current function, and the line $\zeta=$ const. moves with the particles of the fluid.

The study of the integral leads to a curious form for the equation of continuity which is given for motion in three dimensions thus:—

Let $\lambda=$const., $\mu=$const. $\nu=$const., be three surfaces always containing the same particles. The motion of the fluid in the faces of the element of volume included between these three surfaces and the three consecutive surfaces $\lambda+\delta\lambda$, $\mu+\delta\mu$, $\nu+\delta\nu$ is considered. From the fact that the volume is invariable, it follows that

$$\frac{\delta}{\delta\lambda}\left(\frac{\lambda_t}{A}\right)+\frac{\delta}{\delta\mu}\left(\frac{\mu_t}{A}\right)+\frac{\delta}{\delta\nu}\left(\frac{\nu_t}{A}\right)=0,$$

where

$$A=\begin{vmatrix}\lambda_x & \lambda_y & \lambda_z\\ \mu_x & \mu_y & \mu_z\\ \nu_x & \nu_y & \nu_z\end{vmatrix};$$

δ denotes differentiation when λ, μ, ν, t are independent variables; and the suffixes denote differentiation when x, y, z, t are independent variables.

It is next shown that the equation in ζ can be simplified provided that the lines $\zeta=$ const. do not change their form. The result is

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)F(\zeta,t)=\zeta+2\dot{\omega}$$

(provided that ζ be not constant throughout the fluid),

where F is an arbitrary function, and $\dot{\omega}$ the angular velocity of rotation of the lines $\zeta=$ const.

II. *Annular Vortices.*

If $\lambda =$ const. be the surface of an annular vortex, r the distance of a point from the straight axis, z the distance from some fixed plane perpendicular to the straight axis, then must

$$\frac{1}{r^2}\left(\frac{d^2}{dz^2}+\frac{d^2}{dr^2}-\frac{1}{r}\,\frac{d}{dr}\right)\int \delta r\left(\frac{r\lambda_t}{\lambda_z}\right)\begin{matrix} r \\ \lambda \\ t\end{matrix}=\lambda$$

(excepting a particular case).

If λ be of the form $f\left(\dfrac{r-R}{R},\ R^2(z-Z)\right)$, where R, Z are functions of t, so that if the aperture of the ring increases in any ratio, the breadth of the ring in the plane of the aperture increases in the same ratio, but the breadth of the ring parallel to the direction of the straight axis diminishes inversely as the square of the ratio (including the case where the vortex is of invariable form), then the equation becomes

$$\frac{1}{r^2}\left(\frac{d^2}{dz^2}+\frac{d^2}{dr^2}-\frac{1}{r}\,\frac{d}{dr}\right)F(\lambda,\,t)=\lambda$$

(excepting a particular case).

I. *The Differential Equation of a Cylindrical Vortex.*

Let the normal section of the cylinders be taken as the plane of x, y; let the motion be supposed to take place in this plane; let u, v be the velocities parallel to the axes of coordinates at the point x, y at time t. Let p be the pressure, ρ the density supposed constant, and V the potential of the impressed forces.

The equations of motion are

$$\left.\begin{aligned}\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}&=-\frac{d}{dx}\left(\frac{p}{\rho}+V\right)\\ \frac{dv}{dt}+u\frac{dv}{dx}+v\frac{dv}{dy}&=-\frac{d}{dy}\left(\frac{p}{\rho}+V\right)\\ \frac{du}{dx}+\frac{dv}{dy}&=0\end{aligned}\right\} \quad \text{..................(I.)}.$$

From these equations it follows that

$$\left(\frac{d}{dt}+u\,\frac{d}{dx}+v\,\frac{d}{dy}\right)\left(\frac{du}{dy}-\frac{dv}{dx}\right)=0 \quad \text{...........(II.)}.$$

The third equation requires that there should exist a current function Λ such that $$u=\frac{d\Lambda}{dy},\quad v=-\frac{d\Lambda}{dx}.$$

Substituting these values of u and v in (II.), it becomes

$$\left(\frac{d}{dt}+\frac{d\Lambda}{dy}\frac{d}{dx}-\frac{d\Lambda}{dx}\frac{d}{dy}\right)\left(\frac{d^2\Lambda}{dx^2}+\frac{d^2\Lambda}{dy^2}\right)=0 \text{ ....(III.).}$$

Thus the current function satisfies a differential equation of the third order.

If a single symbol ζ be put for $\frac{du}{dy}-\frac{dv}{dx}$, the molecular rotation, so that

$$\zeta=\frac{d^2\Lambda}{dx^2}+\frac{d^2\Lambda}{dy^2} \text{ ....(IV.),}$$

then it is possible to eliminate Λ from (III.) and (IV.).

For, let (IV.) be differentiated with regard to x and y separately; and let (III.) be written

$$\frac{d\zeta}{dt}+\frac{d\Lambda}{dy}\frac{d\zeta}{dx}-\frac{d\Lambda}{dx}\frac{d\zeta}{dy}=0,$$

and then differentiated with regard to x once, y once, x twice, x and y each once, and y twice.

In this manner there will be obtained altogether nine equations, containing ζ and its differential coefficients as far as those of the third order; but the equations are linear in the nine quantities $\frac{d\Lambda}{dx}$, $\frac{d\Lambda}{dy}$, $\frac{d^2\Lambda}{dx^2}$, $\frac{d^2\Lambda}{dx\,dy}$, $\frac{d^2\Lambda}{dy^2}$, $\frac{d^3\Lambda}{dx^3}$, $\frac{d^3\Lambda}{dx^2\,dy}$, $\frac{d^3\Lambda}{dx\,dy^2}$, $\frac{d^3\Lambda}{dy^3}$. Therefore $\frac{d\Lambda}{dx}$ and $\frac{d\Lambda}{dy}$ may be found in terms of ζ and its differential coefficients as far as those of the third order. Then, since $\frac{d}{dy}\left(\frac{d\Lambda}{dx}\right)=\frac{d}{dx}\left(\frac{d\Lambda}{dy}\right)$, ζ satisfies an equation of the fourth order.

But an equation in ζ of a more manageable form may be obtained thus: supposing Λ is not such as to make ζ constant, treat

$$\frac{d\zeta}{dt}+\frac{d\Lambda}{dy}\frac{d\zeta}{dx}-\frac{d\Lambda}{dx}\frac{d\zeta}{dy}=0,$$

as a partial differential equation to find Λ, as if $\frac{d\zeta}{dt}$, $\frac{d\zeta}{dx}$, $\frac{d\zeta}{dy}$ were known functions of x, y, t, and, denoting the differential coefficients of ζ by suffixes, the auxiliary system is

$$\frac{dx}{-\zeta_y}=\frac{dy}{\zeta_x}=\frac{d\Lambda}{-\zeta_t}.$$

As there are no differential coefficients of Λ with regard to t, and

as each member of the auxiliary system $=\dfrac{\zeta_x dx+\zeta_y dy}{0}$, it follows that $\zeta=$ const. is an integral of the system. To obtain a second integral, express y as a function of x, ζ, t, and substitute in the auxiliary system, then another integral is

$$\Lambda-\int\frac{\zeta_t}{\zeta_y}\,dx=\text{const.}$$

The complete integral may then be expressed fully thus:

$$\Lambda=\int\delta x\left(\frac{\zeta_t}{\zeta_y}\right)\overset{x}{\underset{t}{\zeta}}+F(\zeta,\,t),$$

where δ denotes differentiation with regard to the system of variables x, ζ, t; and $\int\delta x\left(\frac{\zeta_t}{\zeta_y}\right)\overset{x}{\underset{t}{\zeta}}$ denotes that $\frac{\zeta_t}{\zeta_y}$ is to be expressed as a function of x, ζ, t by elimination of y, and the result integrated with regard to x, treating ζ, t as constants.

[The above form of Λ may be easily shown to satisfy the differential equation (III.), whether the process adopted in obtaining it has resulted in the most general integral or not, thus:

$$\frac{d\Lambda}{dx}=\frac{\delta\Lambda}{\delta x}+\frac{\delta\Lambda}{\delta\zeta}\frac{d\zeta}{dx},$$

$$\frac{d\Lambda}{dy}=\qquad\frac{\delta\Lambda}{\delta\zeta}\frac{d\zeta}{dy}$$

therefore $$\frac{d\Lambda}{dy}\frac{d\zeta}{dx}-\frac{d\Lambda}{dx}\frac{d\zeta}{dy}=-\frac{d\zeta}{dy}\frac{\delta\Lambda}{\delta x}=-\frac{d\zeta}{dy}\frac{\zeta_t}{\zeta_y},$$

therefore $$\frac{d\zeta}{dt}+\frac{d\Lambda}{dy}\frac{d\zeta}{dx}-\frac{d\Lambda}{dx}\frac{d\zeta}{dy}=0,$$

which is (III.).]

Substituting the value found for Λ in (IV.),

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\left[\int\delta x\left(\frac{\zeta_t}{\zeta_y}\right)\overset{x}{\underset{t}{\zeta}}+F(\zeta,\,t)\right]=\zeta.$$

Observing that the arbitrary function $F(\zeta,t)$ is necessarily contained in the integral, the equation may be written more concisely

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\int\delta x\left(\frac{\zeta_t}{\zeta_y}\right)\overset{x}{\underset{t}{\zeta}}=\zeta\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(V.)}.$$

(It should be remembered that this form has been obtained on the hypothesis that ζ is not constant.)

The form of this equation is interesting. The line $\zeta = \text{const.}$ always contains the same particles of fluid. Suppose that at the time t the point x, y is on it; and at time $t+\delta t$ the point x, $y+\delta y$ is on it,

then $$\zeta(x, y, t) = \zeta(x, y+\delta y, t+\delta t),$$

therefore $$0 = \zeta_y \delta y + \zeta_t \delta t,$$

therefore $$\delta y = -\frac{\zeta_t}{\zeta_y}\delta t.$$

If now S be the area included between a small portion ϵ of the axis of x, the ordinates at its extremities, and the line $\zeta = \text{const.}$ at time t; and if $S+\delta S$ be the area at time $t+\delta t$; then

$$\delta S = -\epsilon \frac{\zeta_t}{\zeta_y}\delta t.$$

If δs be the small arc of the line $\zeta = \text{const.}$ whose projection is ϵ, and if the positive direction of the arc be such that the direction of motion of $\zeta = \text{const.}$ is across it from left to right, then $\epsilon = -\delta x$,

therefore $$\delta S = \frac{\zeta_t}{\zeta_y}\delta x\, \delta t.$$

This is the same as the flux of fluid across the line fixed in space with which δs coincides at time t, since $\zeta = \text{const.}$ moves with the particles of fluid; therefore flux in unit time $= \frac{\zeta_t}{\zeta_y}\delta x$; therefore the current function $= \int \frac{\zeta_t}{\zeta_y}\delta x$, where the integration must be extended along the line $\zeta = \text{const.}$, or, in the notation adopted above,

the current function $$= \int \delta x \left(\frac{\zeta_t}{\zeta_y}\right)\begin{matrix} x \\ \zeta \\ t \end{matrix}.$$

This coincides with the result obtained by the method adopted for integrating the partial differential equation (III.).

The same form might be obtained by considering the flux of fluid normal to the arc δs, and, as there is no advantage in confining the discussion to motion in two dimensions, the question may be generalised thus:—

Let $\lambda = \text{const.}$, $\mu = \text{const.}$, $\nu = \text{const.}$ be three surfaces which always contain the same particles of fluid. Then the volume described

by an element of area δS situated on $\lambda =$ const. in unit time

$$= (\text{normal velocity})\ \delta S,$$

$$= \frac{u\lambda_x + v\lambda_y + w\lambda_z}{\sqrt{\lambda_x^2+\lambda_y^2+\lambda_z^2}}\,\delta S = \frac{-\lambda_t\,\delta S}{\sqrt{\lambda_x^2+\lambda_y^2+\lambda_z^2}},$$

and this may be expressed in any of the equivalent forms

$$\frac{-\lambda_t\,\delta y\,\delta z}{\lambda_x} = \frac{-\lambda_t\,\delta z\,\delta x}{\lambda_y} = \frac{-\lambda_t\,\delta x\,\delta y}{\lambda_z}.$$

Now consider the element of volume contained between the six surfaces λ, μ, ν, $\lambda+\delta\lambda$, $\mu+\delta\mu$, $\nu+\delta\nu$.

Then δS (normal distance between λ and $\lambda+\delta\lambda$)

$$= \text{volume} = \frac{\delta\lambda\ \ \delta\mu\ \ \delta\nu}{\begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ \mu_x & \mu_y & \mu_z \\ \nu_x & \nu_y & \nu_z \end{vmatrix}},$$

and (normal distance between λ and $\lambda+\delta\lambda$)

$$= \frac{\delta\lambda}{\sqrt{\lambda_x^2+\lambda_y^2+\lambda_z^2}},$$

whence
$$\delta S = \frac{\sqrt{\lambda_x^2+\lambda_y^2+\lambda_z^2}\,.\,\delta\mu\,.\,\delta\nu}{\begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ \mu_x & \mu_y & \mu_z \\ \nu_x & \nu_y & \nu_z \end{vmatrix}};$$

therefore volume described by δS

$$= \frac{-\lambda_t\,.\,\delta\mu\,.\,\delta\nu}{\begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ \mu_x & \mu_y & \mu_z \\ \nu_x & \nu_y & \nu_z \end{vmatrix}}.$$

If the determinant be replaced by A, and δS be made in succession the six faces of the element of volume, then, since the change in the volume is zero,

$$\frac{\delta}{\delta\lambda}\left(\frac{\lambda_t}{A}\right) + \frac{\delta}{\delta\mu}\left(\frac{\mu_t}{A}\right) + \frac{\delta}{\delta\nu}\left(\frac{\nu_t}{A}\right) = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots(\text{A}),$$

where δ denotes differentiation when the independent variables are λ, μ, ν, t.

To verify this result, it will be reduced to a known form of the equation of continuity.

The following equations are known to be true:

$$\frac{d}{dx} = \lambda_x \frac{\delta}{\delta\lambda} + \mu_x \frac{\delta}{\delta\mu} + \nu_x \frac{\delta}{\delta\nu},$$

$$\frac{d}{dy} = \lambda_y \frac{\delta}{\delta\lambda} + \mu_y \frac{\delta}{\delta\mu} + \nu_y \frac{\delta}{\delta\nu},$$

$$\frac{d}{dz} = \lambda_z \frac{\delta}{\delta\lambda} + \mu_z \frac{\delta}{\delta\mu} + \nu_z \frac{\delta}{\delta\nu}.$$

Now let the reciprocal determinant of A be

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

then
$$A\frac{\delta}{\delta\lambda} = a_{11}\frac{d}{dx} + a_{12}\frac{d}{dy} + a_{13}\frac{d}{dz},$$

$$A\frac{\delta}{\delta\mu} = a_{21}\frac{d}{dx} + a_{22}\frac{d}{dy} + a_{23}\frac{d}{dz},$$

$$A\frac{\delta}{\delta\nu} = a_{31}\frac{d}{dx} + a_{32}\frac{d}{dy} + a_{33}\frac{d}{dz}.$$

Therefore it is necessary to show that

$$\frac{a_{11}\lambda_{xt} + a_{12}\lambda_{yt} + a_{13}\lambda_{zt}}{A} + \lambda_t\left(a_{11}\frac{d}{dx} + a_{12}\frac{d}{dy} + a_{13}\frac{d}{dz}\right)\frac{1}{A}$$
$$+ \frac{a_{21}\mu_{xt} + a_{22}\mu_{yt} + a_{23}\mu_{zt}}{A} + \mu_t\left(a_{21}\frac{d}{dx} + a_{22}\frac{d}{dy} + a_{23}\frac{d}{dz}\right)\frac{1}{A}$$
$$+ \frac{a_{31}\nu_{xt} + a_{32}\nu_{yt} + a_{33}\nu_{zt}}{A} + \nu_t\left(a_{31}\frac{d}{dx} + a_{32}\frac{d}{dy} + a_{33}\frac{d}{dz}\right)\frac{1}{A} = 0,$$

i.e.,
$$\frac{1}{A}\frac{dA}{dt} + (a_{11}\lambda_t + a_{21}\mu_t + a_{31}\nu_t)\frac{d}{dx}\left(\frac{1}{A}\right)$$
$$+ (a_{12}\lambda_t + a_{22}\mu_t + a_{32}\nu_t)\frac{d}{dy}\left(\frac{1}{A}\right)$$
$$+ (a_{13}\lambda_t + a_{23}\mu_t + a_{33}\nu_t)\frac{d}{dz}\left(\frac{1}{A}\right) = 0.$$

But since
$$\lambda_t+u\lambda_x+v\lambda_y+w\lambda_z = 0,$$
$$\mu_t+u\mu_x+v\mu_y+w\mu_z = 0,$$
$$\nu_t+u\nu_x+v\nu_y+w\nu_z = 0;$$

solving for u, v, w, and substituting in the equation to be verified, it

becomes
$$\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}+w\frac{d}{dz}\right)\frac{1}{A} = 0,$$

therefore
$$\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}+w\frac{d}{dz}\right)\begin{vmatrix}\lambda_x & \lambda_y & \lambda_z\\ \mu_x & \mu_y & \mu_z\\ \nu_x & \nu_y & \nu_z\end{vmatrix} = 0,$$

which is equivalent to what is called the integral equation of continuity in Thomson and Tait's "Natural Philosophy," Vol. I., Part I., Section 192.

Since the flux of fluid across that part of the surface $\lambda = \text{const.}$, which is cut out by the surfaces μ, ν, $\mu+\delta\mu$, $\nu+\delta\nu$, is $\frac{-\lambda_t\,\delta\mu\,\delta\nu}{A}$, the investigation seems to point to the existence of functions analogous to the current function, one for each of the surfaces λ, μ, ν. Calling them L, M, N,

$$\frac{\delta^2 L}{\delta\mu\,\delta\nu} = \frac{-\lambda_t}{A},\quad \frac{\delta^2 M}{\delta\nu\,\delta\lambda} = \frac{-\mu_t}{A},\quad \frac{\delta^2 N}{\delta\lambda\,\delta\mu} = \frac{-\nu_t}{A}.$$

The only property which the writer has succeeded in obtaining regarding them is that which follows from the new form of the equation of continuity given in this paper. It is this:

$$\frac{\delta^3}{\delta\lambda\,\delta\mu\,\delta\nu}(L+M+N) = 0 \quad \text{.....................(B).}$$

Returning now to the equations in Λ and ζ, they may both be very greatly simplified if the stream lines do not change their form during the motion. This is expressed by putting

$$\Lambda = f(x', y'),$$

where
$$x' = (x-X)\cos\omega+(y-Y)\sin\omega,$$
$$y' = -(x-X)\sin\omega+(y-Y)\cos\omega,$$

where X, Y, ω are functions of t only.

$$\frac{d}{dx} = \cos\omega \frac{d}{dx'} - \sin\omega \frac{d}{dy'},$$

$$\frac{d}{dy} = \sin\omega \frac{d}{dx'} + \cos\omega \frac{d}{dy'},$$

therefore
$$\frac{d^2}{dx^2} + \frac{d^2}{dy^2} = \frac{d^2}{dx'^2} + \frac{d^2}{dy'^2}.$$

$$\frac{d}{dt} = \left\{-(x-X)\sin\omega\,.\,\dot\omega + (y-Y)\cos\omega\,.\,\dot\omega - \dot X\cos\omega - \dot Y\sin\omega\right\}\frac{d}{dx'}$$

$$+\left\{-(x-X)\cos\omega\,.\,\dot\omega - (y-Y)\sin\omega\,.\,\dot\omega + \dot X\sin\omega - \dot Y\cos\omega\right\}\frac{d}{dy'}$$

$$= (\dot\omega y' - \dot X\cos\omega - \dot Y\sin\omega)\frac{d}{dx'} + (-\dot\omega x' + \dot X\sin\omega - \dot Y\cos\omega)\frac{d}{dy'},$$

therefore
$$\frac{d}{dt} + \frac{d\Lambda}{dy}\frac{d}{dx} - \frac{d\Lambda}{dx}\frac{d}{dy}$$

$$= \left(\dot\omega y' - \dot X\cos\omega - \dot Y\sin\omega + \frac{d\Lambda}{dy'}\right)\frac{d}{dx'}$$

$$-\left(\dot\omega x' - \dot X\sin\omega + \dot Y\cos\omega + \frac{d\Lambda}{dx'}\right)\frac{d}{dy'}$$

$$= \frac{d}{dy'}\left\{\Lambda + \frac{\dot\omega}{2}(x'^2+y'^2) + x'(-\dot X\sin\omega + \dot Y\cos\omega) - y'(\dot X\cos\omega + \dot Y\sin\omega)\right\}\frac{d}{dx'}$$

$$-\frac{d}{dx'}\left\{\Lambda + \frac{\dot\omega}{2}(x'^2+y'^2) + x'(-\dot X\sin\omega + \dot Y\cos\omega) - y'(\dot X\cos\omega + \dot Y\sin\omega)\right\}\frac{d}{dy'}$$

$$= \frac{d\psi}{dy'}\frac{d}{dx'} - \frac{d\psi}{dx'}\frac{d}{dy'},$$

if $\psi = \Lambda + \frac{\dot\omega}{2}(x'^2+y'^2) + x'(-\dot X\sin\omega + \dot Y\cos\omega) - y'(\dot X\cos\omega + \dot Y\sin\omega)$.

Therefore (III.) becomes

$$\frac{d\psi}{dy'}\frac{d}{dx'}\left(\frac{d^2\Lambda}{dx'^2} + \frac{d^2\Lambda}{dy'^2}\right) - \frac{d\psi}{dx'}\frac{d}{dy'}\left(\frac{d^2\Lambda}{dx'^2} + \frac{d^2\Lambda}{dy'^2}\right) = 0,$$

therefore
$$\psi = F\left(\frac{d^2\Lambda}{dx'^2} + \frac{d^2\Lambda}{dy'^2},\ t\right).$$

Returning to the original system of variables

$$\psi = \Lambda + \frac{\omega}{2}(x^2+y^2) - \dot{X}(y-Y) + \dot{Y}(x-X),$$

$$\frac{d^2\Lambda}{dx'^2} + \frac{d^2\Lambda}{dy'^2} = \frac{d^2\Lambda}{dx^2} + \frac{d^2\Lambda}{dy^2} = \zeta$$

$$= \frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} - 2\dot{\omega},$$

$$\psi = F(\zeta, t);$$

therefore $$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right) F(\zeta, t) = \zeta + 2\dot{\omega} \quad \text{.................(VI.)},$$

or else $\zeta =$ const., when the current function is to be obtained from (IV.).

Exactly the same result would have been obtained if the investigation had been based on the equation in ζ in the form (V.).

II. *The Differential Equation of an Annular Vortex.*

Let the straight axis of the annular vortices be the axis of z, let r be the distance of any point from the straight axis, with regard to which the motion is supposed to be symmetrical.

Let τ, w be the velocities in the directions of r, z respectively; then

$$\left.\begin{aligned} \frac{d\tau}{dt} + \tau\frac{d\tau}{dr} + w\frac{d\tau}{dz} &= -\frac{d}{dr}\left(\frac{p}{\rho} + V\right) \\ \frac{dw}{dt} + \tau\frac{dw}{dr} + w\frac{dw}{dz} &= -\frac{d}{dz}\left(\frac{p}{\rho} + V\right) \\ \frac{d}{dr}(r\tau) + \frac{d}{dz}(rw) &= 0 \end{aligned}\right\} \quad \text{..............(I.')}.$$

Whence $$\left(\frac{d}{dt} + \tau\frac{d}{dr} + w\frac{d}{dz}\right)\left(\frac{\frac{d\tau}{dz} - \frac{dw}{dr}}{r}\right) = 0 \quad \text{..............(II.')}.$$

The third of the equations (I.') requires the existence of a function Λ, such that $$\tau = \frac{1}{r}\frac{d\Lambda}{dz}, \quad w = -\frac{1}{r}\frac{d\Lambda}{dr}.$$

Substituting in (II.'), it becomes

$$\left(\frac{d}{dt} + \frac{1}{r}\frac{d\Lambda}{dz}\frac{d}{dr} - \frac{1}{r}\frac{d\Delta}{dr}\frac{d}{dz}\right)\left\{\frac{1}{r^2}\left(\frac{d^2\Lambda}{dz^2} + \frac{d^2\Delta}{dr^2} - \frac{1}{r}\frac{d\Lambda}{dr}\right)\right\} = 0 \ldots\text{(III.')}.$$

Put $$\frac{1}{r^2}\left(\frac{d^2\Lambda}{dz^2}+\frac{d^2\Lambda}{dr^2}-\frac{1}{r}\frac{d\Lambda}{dr}\right)=\lambda \quad\text{.................(IV.')}.$$

Then (III.') may be written

$$\lambda_t+\frac{\lambda_r}{r}\frac{d\Lambda}{dz}-\frac{\lambda_z}{r}\frac{d\Lambda}{dr}=0;$$

therefore, supposing Λ is not such as to make λ constant, it follows

that $$\Lambda=\int\delta r\left(\frac{r\lambda_t}{\lambda_z}\right)\begin{matrix}r\\ \lambda\\ t\end{matrix}+F(\lambda,t),$$

therefore $$\frac{1}{r^2}\left(\frac{d^2}{dz^2}+\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right)\int\delta r\left(\frac{r\lambda_t}{\lambda_z}\right)\begin{matrix}r\\ \lambda\\ t\end{matrix}=\lambda \quad\text{............(V.')},$$

remembering that the arbitrary function is contained in the integral.

If the ring-shaped surfaces $\lambda=$ const., do not alter their form throughout the motion, $\lambda=f(r-R,\ z-Z)$, where R is constant, and Z a function of the time only. In this case,

$$\int\delta r\left(\frac{r\lambda_t}{\lambda_z}\right)\begin{matrix}r\\ \lambda\\ t\end{matrix}=\int\delta r\,(-r\dot{Z})\begin{matrix}r\\ \lambda\\ t\end{matrix}=-\frac{r^2}{2}\dot{Z}+F(\lambda\ \ t).$$

A somewhat more general case is obtained by taking

$$\lambda=f\left(\frac{r-R}{R},\ R^2(z-Z)\right),$$

where R, Z are functions of t. This will give annular vortices of invariable volume, for, when the circular axis of the ring increases in any ratio, the breadth of the ring in the direction of r increases in the same ratio, whilst its breadth in the direction of z diminishes inversely as the square of the ratio; hence, by Guldinus' theorem, the volume is unaltered.

Put for brevity

$$\frac{r-R}{R}=\xi,\quad R^2(z-Z)=\eta,$$

$$\therefore\quad \int\delta r\left(\frac{r\lambda_t}{\lambda_z}\right)\begin{matrix}r\\ \lambda\\ t\end{matrix}=\int\delta r\,.\,r\left\{\frac{\frac{\delta f}{\delta\xi}\left(-\frac{r}{R^2}\dot{R}\right)+\frac{\delta f}{\delta\eta}\left[2R\dot{R}(z-Z)-R^2\dot{Z}\right]}{\frac{\delta f}{\delta\eta}R^2}\right\}\begin{matrix}r\\ \lambda\\ t\end{matrix}$$

$$=\int\delta r,\left\{2r(z-Z)\frac{\dot{R}}{R}-r\dot{Z}-\frac{\dot{R}}{R^4}r^2\frac{\frac{\delta f}{\delta\xi}}{\frac{\delta f}{\delta\eta}}\right\}\begin{matrix}r\\ \lambda\\ t\end{matrix}.$$

In performing the integration, z is to be first expressed as a function of r, λ, t by means of the equation

$$\lambda = f\left(\frac{r-R}{R},\ R^2(z-Z)\right),$$

so that λ, r, t are independent, and z the dependent variable. Differentiating with regard to r,

$$0 = \frac{\delta f}{\delta \xi}\frac{1}{R} + \frac{\delta f}{\delta \eta} R^2 \frac{\delta z}{\delta r},$$

therefore $$\int \delta r \left(\frac{r\lambda_t}{\lambda_z}\right)_{t}^{r} \overset{r}{\underset{t}{\lambda}} = \int \delta r \left\{ 2r(z-Z)\frac{\dot{R}}{R} - r\dot{Z} + \frac{\dot{R}}{R} r^2 \frac{\delta z}{\delta r} \right\} \overset{r}{\underset{t}{\lambda}}$$

$$= \frac{\dot{R}}{R} r^2 (z-Z) - \frac{r^2}{2}\dot{Z} + F(\lambda, t).$$

Putting $\dot{R} = 0$, this reduces to the case where the form of the vortex is invariable.

Therefore the equation is

$$\frac{1}{r^2}\left(\frac{d^2}{dz^2} + \frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right)\left\{F(\lambda, t) + \frac{\dot{R}}{R} r^2 (z-Z) - \frac{r^2}{2}\dot{Z}\right\} = \lambda,$$

i.e., $$\frac{1}{r^2}\left(\frac{d^2}{dz^2} + \frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right) F(\lambda, t) = \lambda \quad \text{..............(VI.')},$$

or else $\lambda = \text{const.}$, when the current function is to be obtained from (IV'.).

Note.—Some particular cases have been fully worked out in a paper by the writer entitled, "On the motion of fluid part of which is moving rotationally and part irrotationally," published in the *Philosophical Transactions of the Royal Society*, Part II., 1884.

February 12th, 1885.

J. W. L. GLAISHER, Esq., F.R.S., President, in the Chair.

Miss Emily Perrin, of Cheltenham Ladies' College (late of Girton College), was elected a member, and Mr. J. Griffiths was admitted into the Society.

Mr. Tucker read the following papers:—

Sur les figures semblablement variables: Prof. J. Neuberg, of Liège;

On the Extension of Ivory's and Jacobi's Distance-Correspondences for Quadric Surfaces: Prof. J. Larmor; and

Some Properties of a Quadrilateral in a Circle the Rectangles under whose Opposite Sides are equal: Mr. Tucker.

Messrs. Jenkins and S. Roberts spoke on the subject of Prof. Neuberg's paper.

The following presents were received:—

Cabinet likeness of Mr. W. S. B. Woolhouse, and two photographs of geometrical figures from Mr. Woolhouse.

"Educational Times," for February.

"Physical Society of London—Proceedings," Vol. vi., Part iii., September to December, 1884.

"Johns Hopkins University Circulars," Vol. iv., No. 36, January, 1885.

"Report of the Superintendent of the United States Naval Observatory," for the year ending October 30th, 1884, 8vo; Washington, 1884.

"Bulletin de la Société Mathématique de France," Tome xii., No. 5; Paris, 1884.

"Atti della R. Accademia dei Lincei," Serie quarta, Vol. i., F. 1, 2, and 3; Roma, 1884, 1885.

"Journal für die reine und angewandte Mathematik," B. xcviii., H. 1; Berlin, 1885.

"Annales de l'Ecole Polytechnique de Delft," 1re liv., 4to; Leide, 1884.

"Bulletin des Sciences Mathématiques et Astronomiques," T. ix., Jan. 1885; Paris.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. ix., St. 1.; Leipzig, 1885.

"Tesis leída en el Examen professional de ingeniero geógrafo," por J. de Mendizabal Tamborrel, 8vo; Mexico, 1884.

"Sitzungsberichte der physikalisch-medicinischen Societät zu Erlangen," 16 Heft, Oktober 1882—Oktober 1884.

"American Journal of Mathematics," Vol. vii., No. 2.

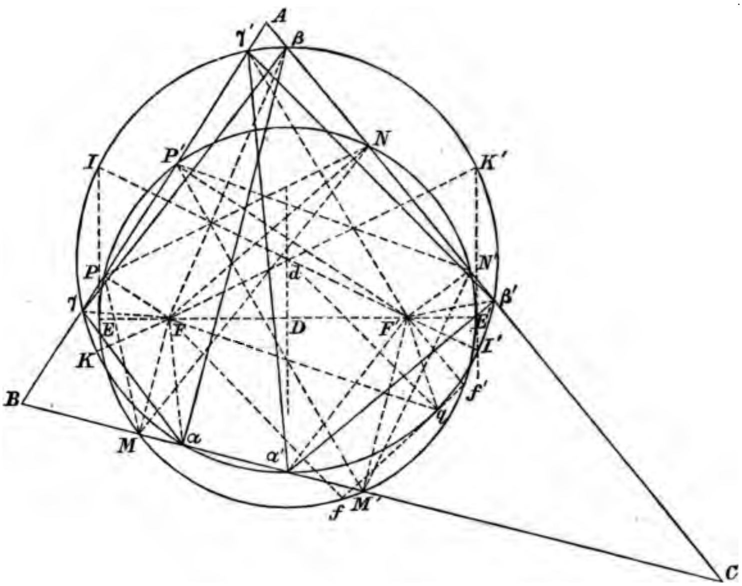
Sur les figures semblablement variables. *Par* M. J. Neuberg, Professeur à l'Université de Liège.

[*Read February* 12*th*, 1885.]

M. H. M. Taylor, dans les *Proc. of the Lond. Math. Soc.*, Vol. xv., p. 122, a signalé des relations fort intéressantes entre les intersections d'un cercle avec les côtés d'un triangle. La présente Note a pour objet de préciser et de compléter ces relations en les rattachant à la

théorie des figures semblablement variables et à celle des podaires obliques d'un foyer d'une conique.

1. Soit $\alpha\beta\gamma$ un triangle variable sous les conditions de rester constant de forme et inscrit au triangle fixe ABC. Les cercles circon-



scrits aux triangles $A\beta\gamma$, $B\gamma\alpha$, $C\alpha\beta$ se coupent en un même point F, tel que $\quad \beta F\gamma = \pi - A, \quad \gamma F\alpha = \pi - B, \quad \alpha F\beta = \pi - C,$

$$BFC = BF\alpha + CF\alpha = B\gamma\alpha + C\beta\alpha = \pi - B - B\alpha\gamma + \pi - C - C\alpha\beta = A + \alpha,$$

$$AFC = B + \beta, \quad AFB = C + \gamma.$$

Ces égalités montrent que le point F est fixe par rapport au triangle donné ABC et par rapport au triangle mobile $\alpha\beta\gamma$.

Donc, *lorsqu'un triangle $\alpha\beta\gamma$ reste semblable à lui-même et inscrit à un triangle fixe ABC, il existe un point du triangle mobile qui est fixe ; ce point est celui d'où l'on voit les côtés de $\alpha\beta\gamma$ sous des angles supplémentaires de ceux de ABC, et les côtés de ABC sous des angles respectivement égaux aux sommes des angles correspondants des deux triangles.**

2. Menons FM, FN, FP perpendiculaires à BC, CA, AB. Le tri-

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* Voir *Nouvelles Annales*, t. XVII., p. 48 ; *Nouvelle Correspondance*, t. VI., pp. 65, 72, 219, 321 ; *Mathesis*, t. I., p. 106.

angle MNP est une position particulière de $\alpha\beta\gamma$; c'est le minimum de $\alpha\beta\gamma$.

Soit θ l'angle $\alpha FM = \beta FN = \gamma FP$. Si l'on fait tourner le système des trois droites FM, FN, FP autour de F de l'angle θ et qu'on prenne les intersections avec BC, CA, AB, on aura les sommets du triangle $\alpha\beta\gamma$.

Plus généralement, étant tracées une figure S dans le plan MNP et la figure semblable σ dans le plan $\alpha\beta\gamma$, on transforme S en σ en faisant tourner S de l'angle θ autour de F et en modifiant les distances des points de S au point F dans le rapport $\cos\theta : 1$.

Autrement dit, on passe de S à σ en construisant, sur les droites joignant F aux différents points de S, des triangles rectangles semblables ayant en F le même angle θ.

De là résulte que, si D et d sont deux points homologues par rapport à MNP et $\alpha\beta\gamma$, le triangle dFD est semblable à αFM. Donc tout point du plan mobile MNP décrit une droite. En particulier, si D est le centre du cercle MNP, le centre d du cercle $\alpha\beta\gamma$ décrit une droite perpendiculaire à FD.

F est un *centre permanent de similitude* du triangle mobile $\alpha\beta\gamma$.

Deux lignes homologues quelconques des figures S et σ sont dans le rapport $\cos\theta : 1$.

3. On sait qu'une tangente quelconque à une parabole, limitée à deux tangentes fixes, est vue du foyer sous un angle supplémentaire de celui des tangentes fixes ; la réciproque de ce théorème est également vraie. Si on applique cette proposition aux angles MFN, NFP, PFM tournant autour de F, on voit que *les côtés $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ du triangle mobile $\alpha\beta\gamma$ enveloppent trois paraboles touchant deux côtés du triangle ABC et ayant pour foyer commun le point F;* les sommets de ces courbes sont les projections de F sur les lignes NP, PM, MN.

Plus généralement, toute droite de la figure S enveloppe une parabole de foyer F; car deux points de cette droite décrivent des droites (2°), et la distance de ces points est vue de F sous un angle constant.

4. Le cercle MNP rencontre les côtés de ABC en trois nouveaux points M', N', P', qui sont les projections du point F' symétrique de F par rapport au centre D. Les points F, F' sont les foyers d'une conique U inscrite au triangle ABC ; le cercle MNP est la podaire de F et F' par rapport à U, le diamètre EFE' du cercle est un axe de U ; le second axe de U est dirigé suivant la droite Dd.

Soient α', β', γ' les intersections de BC, CA, AB avec le cercle $\alpha\beta\gamma$. Le triangle $\alpha'\beta'\gamma'$ reste toujours semblable à lui-même, lorsque le triangle $\alpha\beta\gamma$ est constant de forme, car on a

$$\alpha+\alpha' = B+C, \quad \beta+\beta' = C+A, \quad \gamma+\gamma' = A+B,$$

α, β, etc. désignant les angles des triangles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$. Il est intéressant d'observer que

$$BFC = \alpha + A = \pi - \alpha', \quad CFA = \pi - \beta', \quad AFB = \pi - \gamma',$$

$$BF'C = \alpha' + A = \pi - \alpha, \quad CF'A = \pi - \beta, \quad AF'B = \pi - \gamma.$$

F' est le centre permanent de similitude du triangle $\alpha'\beta'\gamma'$.

Le cercle MNP est le lieu des projections des points F, F' sur les tangentes à la conique U (podaire de F et F'). Si l'on mène par F des droites faisant dans le même sens avec les tangentes à U l'angle $\frac{1}{2}\pi - \theta$, le lieu des extrémités de ces droites sera le cercle $\alpha\beta\gamma$, dont le centre d est sur le second axe de U; l'angle $dFD = \theta$. Mais l'angle $dFD = dF'D$; donc le même cercle $\alpha\beta\gamma\alpha'\beta'\gamma'$ est aussi le lieu des projections de F' sur les tangentes à U, les projetantes faisant avec ces tangentes l'angle $\frac{1}{2}\pi - \theta$.

On peut énoncer autrement ces résultats : *Si les triangles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ tournent respectivement autour de F et F' avec la même vitesse angulaire, mais en sens contraire, leurs sommets sont toujours sur une même circonférence.*

5. Le cercle $\alpha\beta\gamma\alpha'\beta'\gamma'$ a pour diamètres les droites II', KK', qui passent par les foyers F, F' et sont limitées aux tangentes menées par les sommets E, E' de l'ellipse U.

Je dis que ce cercle touche l'ellipse U. En effet, soit dQ l'une des normales menées de d à U; cette droite est la bissectrice de l'angle FQF', et la perpendiculaire ff' à dQ par le point Q est tangente à U. La circonférence circonscrite au triangle QFF' passe nécessairement par d; car les lignes dQ et dD passent toutes deux par le milieu de l'arc sous-tendu par la corde FF' de ce cercle. Il en résulte que

$$FQf = FdD = \frac{\pi}{2} - \theta, \quad F'Qf' = F'dD = \frac{\pi}{2} - \theta\,;$$

donc Q est la projection de F ou F' sur ff', les projetantes faisant l'angle $\frac{\pi}{2} - \theta$ avec ff' ; par suite Q est un point du cercle $\alpha\beta\gamma$.

D'après cela, *la conique U est l'enveloppe du cercle $\alpha\beta\gamma\alpha'\beta'\gamma'$.*

6. Examinons maintenant deux cas particuliers remarquables.

I. Lorsque $\qquad \alpha = A, \quad \beta = B, \quad \gamma = C,$

on a $\qquad \alpha' = \pi - 2A, \quad \beta' = \pi - 2B, \quad \gamma' = \pi - 2C,$

$$BFC = \alpha + A = 2A, \quad BF'C = \alpha' + A = \pi - A, \text{ etc.}$$

Le point F est le centre du cercle circonscrit à ABC et le point de

concours des hauteurs de $\alpha\beta\gamma$; F' est le point de concours des hauteurs de ABC et le centre du cercle inscrit à $\alpha'\beta'\gamma'$. Le cercle MNP est celui des neuf points de ABC.

II. Supposons $\qquad \alpha = B, \quad \beta = C, \quad \gamma = A$;

alors $\qquad \alpha' = C, \quad \beta' = A, \quad \gamma' = B,$

$$BFC = \alpha + A = B + A = \pi - C, \quad BF'C = \alpha' + A = \pi - B, \text{ etc.,}$$

F et F' sont les points de Brocard de ABC; les points de contact de U avec BC, CA, AB sont les pieds des symédianes de ABC. F est aussi un point de Brocard de $\alpha\beta\gamma$, F' un point de Brocard de $\alpha'\beta'\gamma'$.

7. Soient $A'B'C'$ le triangle formé par les droites $\alpha\beta'$, $\beta\gamma'$, $\gamma\alpha'$, et $A''B''C''$ le triangle formé par les droites $\alpha\gamma'$, $\beta\alpha'$, $\gamma\beta'$. M. Taylor a démontré que les directions des côtés de ces triangles sont constantes, et que ces triangles ont, avec ABC, un même centre d'homologie O.

Les distances de O aux côtés de ABC sont inversement proportionnelles aux produits

$$\sin\alpha \sin\alpha', \quad \sin\beta \sin\beta', \quad \sin\gamma \sin\gamma'.$$

Les droites AO, BO, CO rencontrent BC, CA, AB aux points de contact de U; car on trouve ces points en faisant coïncider α avec α', β avec β', γ avec γ'.

Lorsque $\qquad \alpha = B, \quad \beta = C, \quad \gamma = A,$

O est le centre des symédianes de ABC; les côtés de $A'B'C'$ sont parallèles à ceux de ABC; les côtés de $A''B''C''$ sont parallèles aux tangentes menées par A, B, C au cercle ABC.

Lorsque $\qquad \alpha = A, \quad \beta = B, \quad \gamma = C,$

et que M_a, M_b, M_c, H_a, H_b, H_c sont les milieux des côtés et les pieds des hauteurs du triangle ABC, les côtés du triangle $A'B'C'$ sont parallèles aux droites M_bH_c, M_cH_a, M_aH_b, et ceux de $A''B''C''$ sont parallèles aux droites M_cH_b, M_aH_c, M_bH_a. Les distances du point O aux côtés du triangle ABC sont inversement proportionnelles aux produits $\sin A \sin 2A$, $\sin B \sin 2B$, $\sin C \sin 2C$. La ligne AO et le rayon du cercle ABC qui passe par A rencontrent BC à des distances égales de M_a, etc.

Si les triangles $A'B'C'$ ou $A''B''C''$ se réduisent à ce point O, on a le théorème suivant: Par le point O, on mène trois droites parallèles à M_bH_c, M_cH_a, M_aH_b (ou à M_cH_b, M_aH_c, M_bH_a); les points d'intersection de ces droites avec les côtés du triangle ABC sont situés, six sur une circonférence et trois sur une droite.

On the Extension of Ivory's and Jacobi's Distance-Correspondences for Quadric Surfaces. By J. LARMOR.

[*Read February 12th*, 1885.]

1. The theorem of Ivory for confocal quadrics,—that, if P, Q be two points on a quadric and P', Q' the corresponding points on a confocal, then $PQ' = P'Q$,—is of a fundamental character in a general theory of distance relations.

The correspondence between P and P' is determined by their lying on the same orthogonal trajectory of the system of confocals, which is of course a curve of intersection of two confocals of the other species. The correspondence is linear, each principal coordinate being proportional to the related semi-axis of its surface.

By means of the theorem, Ivory established his very general result relating to the attractions of solids bounded by quadric surfaces, which is true, as Poisson pointed out, for all laws of attraction depending only on the distance.

Jacobi's focal relation, which affirms that a quadric may be specified as the locus of a point whose distances from any three points on a focal conic are respectively equal to the distances of any arbitrary point in a plane from three fixed points in that plane, also flows at once from Ivory's theorem (*cf.* Salmon's *Geometry of Three Dimensions*). But it is important to remark that the proposition requires *equality* in the corresponding distances. If, as we sometimes find, it is only postulated that the same relations exist between the one set of distances as between the other set, *i.e.*, if only *proportionality* is required, there is no longer any locus. To obtain a locus, the comparison must then be instituted between four pairs of distances, as in the investigation given below.

Confocal hyperboloids of one sheet may be considered as generated by two systems of mutually intersecting straight lines. On different confocals these lines clearly correspond to one another, as also do their points of intersection; and their corresponding segments are equal. So that in fact, if the lines were two systems of jointed rods, they could be deformed into a confocal without straining.* Now, if we take $ABCD$ a quadrilateral on the surface of a hyperboloid, and $A'B'C'D'$

* The relations of this system, and of the corresponding system of three dimensions, which proves to be also flexible, are worked out in a paper in the *Proceedings of the Cambridge Philosophical Society*, 1884, Vol. v., Part 2.

the corresponding one on a confocal, we see that corresponding sides are equal, and that there are also six relations of the form $AB' = A'B$, $AC' = A'C$, and so on. Thus we have a prismoidal figure bounded by *gauche* quadrilateral faces such that the corresponding sides of two opposite faces are equal, and the diagonals of any of the six faces or diagonal sections connecting them are also equal: it is easy to trace out in this way its relations, and also to see that any further relation of equality would make it a regular prism.

Again, we may apply Ivory's theorem to the focal ellipse and hyperbola of the confocal system: we thus come directly to the result that the distances of a variable point on either of these curves from any two fixed points on the other are equal to the distances of a variable point on a straight line from any two fixed points on that line. This is Dupin's theorem that each of these conics is the locus of the foci in space of the other, and that any two points on the locus possess the focal property.

These examples illustrate the fundamental character of the theorem. The subject is, however, still treated as a special property of confocal quadrics, rather than as a part of a theory of distances in general. The object of this paper is to discuss the question from the latter point of view.* It will be shown that a similar theory applies to a system of confocal cyclides. But it will also be seen that these are the only systems of surfaces for which such relations are true.

2. Let 2, 3, 4 represent fixed points, and 1 any point in their plane, and let 23 represent the distance between 2 and 3; the relation satisfied by the distances of 1 from 2, 3, 4 is well known and easily

* Sections 7 (α) and 9 (α) have been added since the paper was read. It was not, however, until it had been completed, that I was able to see Darboux's Memoir "Sur les Théorèmes d'Ivory, relatifs aux surfaces homofocales du second degré," *Mémoires de Bordeaux*, T. VIII., pp. 197—280. In this paper, Jacobi's transformation is discussed and applied in an elegant and elaborate manner; so that the statement made above must be limited. Much of the results of the discussion given here is to be found in this Memoir, but the methods followed in it are usually different, and sometimes more analytical and special. On the other hand, some of the results given in this paper, such as (4), (10) and (11), lend themselves easily to an extension of Darboux's theory. Of the problem of §8, a posthumous solution of Jacobi's for the case of conics has, it appears, been published by Hermes, and Darboux gives two solutions for the general case of quadrics. These are both different from the one indicated in §8, which possesses the advantage that it can be at once extended to the case of cyclides.

M. Darboux considers also the generalization of the theory, in which the square of the distance of the point in question from a fixed point is replaced by the square of the length of the tangent drawn from it to a fixed sphere, *i.e.*, by the *power* of the sphere. It may be remarked that all that is given here may be at once extended to that case by making use of the well-known relations between the lengths of such tangents, which are identical in form with (2) and (3) (Salmon's *Conic Sections*, §132*a*, Ex. 4), and o[illegible]her similar relation which corresponds to (4).

found, and is

$$\Sigma_4 12^2 . 24^2 . 41^2 = \Sigma_6 34^2 (12^2 . 34^2 - 13^2 . 24^2 - 14^2 . 23^2) \ldots\ldots (1).$$

We may invert this relation with respect to a point 6; this is done by writing for 12 say, the expression $\dfrac{12}{61 . 62}$; and the result is

$$\Sigma_4 12^2 . 24^2 . 41^2 . 63^4 = \Sigma_6 61^2 . 62^2 . 34^2 (12^2 . 34^2 - 13^2 . 24^2 - 14^2 . 23^2) \ldots (2).$$

We have here a relation connecting the ratios of the distances of any point 6 on a plane from four fixed points 1, 2, 3, 4 on the plane.

If we take the centre of inversion 6 to be out of the original plane, we see that the same relation is true also when the five points are on the surface of a sphere.

3. This relation between the ratios of the distances of a point in a plane from four fixed points in the plane, as well as the corresponding theorem in space for five points, also flows readily from Prof. Cayley's method of multiplication of matrices (Salmon, *Higher Algebra*, Ch. III., Ex. 7, 8). Thus, if $x_r\, y_r\, z_r$ be the coordinates of the point r, and if the point 6 be the origin, and we write down the arrays

$$\begin{vmatrix} 61^2 & x_1 & y_1 & z_1 & 1 \\ 62^2 & x_2 & y_2 & z_2 & 1 \\ 63^2 & x_3 & y_3 & z_3 & 1 \\ 64^2 & x_4 & y_4 & z_4 & 1 \\ 65^2 & x_5 & y_5 & z_5 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix} . \begin{vmatrix} 1 & -2x_1 & -2y_1 & -2z_1 & 61^2 \\ 1 & -2x_2 & -2y_2 & -2z_2 & 62^2 \\ 1 & -2x_3 & -2y_3 & -2z_3 & 63^2 \\ 1 & -2x_4 & -2y_4 & -2z_4 & 64^2 \\ 1 & -2x_5 & -2y_5 & -2z_5 & 65^2 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

their product is zero, as each has one row more than the number of columns; therefore, by the ordinary rule, we have, as Prof. Cayley found,

$$\begin{vmatrix} 0 & 12^2 & 13^2 & 14^2 & 15^2 & 1 \\ 21^2 & 0 & 23^2 & 24^2 & 25^2 & 1 \\ 31^2 & 32^2 & 0 & 34^2 & 35^2 & 1 \\ 41^2 & 42^2 & 43^2 & 0 & 45^2 & 1 \\ 51^2 & 52^2 & 53^2 & 54^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

which is Prof. Cayley's form of Carnot's relation connecting the distances of five points in space.

But if we invert the order of the last rows of the matrices, so that

the units may appear at the other ends, the product gives

$$\begin{vmatrix} 0 & 12^2 & 13^2 & 14^2 & 15^2 & 16^2 \\ 21^2 & 0 & 23^2 & 24^2 & 25^2 & 26^2 \\ 31^2 & 32^2 & 0 & 34^2 & 35^2 & 36^2 \\ 41^2 & 42^2 & 43^2 & 0 & 45^2 & 46^2 \\ 51^2 & 52^2 & 53^2 & 54^2 & 0 & 56^2 \\ 61^2 & 62^2 & 63^2 & 64^2 & 65^2 & 0 \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots (4),$$

which is the relation connecting the ratios of the distances of a variable point, say 6, in space from the other five points,—for it is homogeneous in those distances.

We have seen that it may be deduced from the previous relation by inversion; which is also evident from the present form of expression.

The relations written out at length in § 2, being those connecting the mutual distances of four points in a plane, and connecting the ratios of the distances of a variable point in a plane from four fixed points in it, are obtained at once as above by omitting in the arrays the rows and columns which contain the third dimension z, and the fifth point 5; they are the same determinants as (3) and (4), with a row and column omitted. But to obtain the latter relation, (2), for points on a sphere, the z terms must be retained, and each array becomes an ordinary determinant, with an equal number of rows and columns; it is, however, equal to zero, as the condition that the points 1, 2, 3, 4 lie on a sphere passing through the origin 6.

4. Jacobi's theorem follows, as usual, from (1). If $1'$ be a point whose distances from fixed points $2'$, $3'$, $4'$ are respectively equal to the distances of a point 1 in a plane from fixed points 2, 3, 4 in the same plane, the locus of $1'$ is obtained by substituting $1'2'$, $1'3'$, $1'4'$ for 12, 13, 14 in (1): it is a quadric surface.

Now in the same manner may be investigated the locus of a point $1'$ whose distances from any four fixed points $2'$, $3'$, $4'$, $5'$ are respectively equal to the distances of some point 1 in space from four other fixed points 2, 3, 4, 5. This is obtained by substituting in (3) the distances $1'2'$, $1'3'$, $1'4'$, $1'5'$, instead of 12, 13, 14, 15 in the first row and first column of the determinant, and is

$$\begin{vmatrix} 0 & 1'2'^2 & 1'3'^2 & 1'4'^2 & 1'5'^2 & 1 \\ 2'1'^2 & 0 & 23^2 & 24^2 & 25^2 & 1 \\ 3'1'^2 & 32^2 & 0 & 34^2 & 35^2 & 1 \\ 4'1'^2 & 42^2 & 43^2 & 0 & 45^2 & 1 \\ 5'1'^2 & 52^2 & 53^2 & 54^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0 \ldots\ldots\ldots\ldots (5).$$

The locus is clearly a quadric surface; for, when expressed in Cartesian coordinates, $x^2+y^2+z^2$ is common to each member of the first row and column, and therefore its coefficient in the expanded determinant is a constant, viz., the corresponding minor with sign changed.

But the point 1, with which the correspondence is established, is also confined to a locus in space, which is the quadric

$$\begin{vmatrix} 0 & 12^2 & 13^2 & 14^2 & 15^2 & 1 \\ 21^2 & 0 & 2'3'^2 & 2'4'^2 & 2'5'^2 & 1 \\ 31^2 & 3'2'^2 & 0 & 3'4'^2 & 3'5'^2 & 1 \\ 41^2 & 4'2'^2 & 4'3'^2 & 0 & 4'5'^2 & 1 \\ 51^2 & 5'2'^2 & 5'3'^2 & 5'4'^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0 \;\ldots\ldots\ldots\ldots(6).$$

Now we may take any four points on the quadric locus of 1′, and the four related points on the quadric locus of 1, and determine the corresponding foci for them. These loci will be quadrics, and they will, by hypothesis, pass through 2′, 3′, 4′, 5′ and 2, 3, 4, 5 respectively. We have thus found two pairs of quadrics so related that the distance between any two points on the first pair (one on each) is equal to the distance between the corresponding points on the other pair.

It remains to show that these pairs of quadrics are equal, and may be superposed. We can transfer the points 2, 3, 4, 5 considered as a rigid system into such position that the six relations of the form $23' = 2'3$ are satisfied; for every rigid system possesses six degrees of freedom, which may be accommodated to these conditions. When the points are in this new position, the locus of 1 passes through 2′, 3′, 4′, 5′, and that of 1′ through 2, 3, 4, 5, and these points correspond to one another. The considerations just given then show that these two loci are so related that Ivory's theorem holds, and that they are the only class of surfaces which possess this property.

5. Again, the locus of a point 1′ in a plane whose distances from three fixed points 2′, 3′, 4′ on a line in that plane are proportional to the distances of a variable point 1 on a line from fixed points 2, 3, 4 on that line, is a circular cubic with 2′, 3′, 4′ as foci. For the relation between the ratios of the distances of 1 from 2, 3, 4, is

$$23\,.\,41+34\,.\,21+42\,.\,31=0 \;\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

and therefore the equation of the locus is

$$23\,.\,4'1'+34\,.\,2'1'+42\,.\,3'1'=0 \;\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

The relation (5) is unaltered by inversion; it becomes Ptolemy's theorem for a circle. Hence the locus of a point whose distances from three points are proportional to the distances of a variable point on a circle from three fixed points on it, is a bicircular quartic with those points as foci. But this need not be pursued, as it is a case of what follows.

6. The locus of a point $1'$ whose distances from four fixed points $2'$, $3'$, $4'$, $5'$ are respectively proportional to the distances of a point 1 on a plane (or sphere) from four fixed points 2, 3, 4, 5 on that plane (or sphere), may be determined similarly by aid of (2), or, what is the same, of the relation in a plane which corresponds to (4) in space. Its equation is

$$\begin{vmatrix} 0 & 1'2'^2 & 1'3'^2 & 1'4'^2 & 1'5'^2 \\ 2'1'^2 & 0 & 23^2 & 24^2 & 25^2 \\ 3'1'^2 & 32^2 & 0 & 34^2 & 35^2 \\ 4'1'^2 & 42^2 & 43^2 & 0 & 45^2 \\ 5'1'^2 & 52^2 & 53^2 & 54^2 & 0 \end{vmatrix} = 0 \quad \text{..............(9).}$$

When expressed in Cartesian coordinates, the only terms which contain the running coordinates are those of the first row and column, each of which is of the form $x^2+y^2+z^2+$terms of lower orders. The locus is usually a surface of the fourth order, and, as the imaginary circle at infinity is a nodal curve on the locus, it is a *cyclide*. But a reduction takes place when the points 2, 3, 4, 5 lie in a plane; for the coefficient of $(x^2+y^2+z^2)^2$, being the minor determinant formed by omitting the first row and column, is then zero by (3), and the locus is a cyclide of the third order.

Again, the locus of a point $1'$ whose distances from five fixed points $2'$, $3'$, $4'$, $5'$, $6'$ in space are respectively proportional to the distances of a variable point 1 in space from five other points 2, 3, 4, 5, 6, is determined by aid of (4), and its equation is

$$\begin{vmatrix} 0 & 1'2'^2 & 1'3'^2 & 1'4'^2 & 1'5'^2 & 1'6'^2 \\ 2'1'^2 & 0 & 23^2 & 24^2 & 25^2 & 26^2 \\ 3'1'^2 & 32^2 & 0 & 34^2 & 35^2 & 36^2 \\ 4'1'^2 & 42^2 & 43^2 & 0 & 45^2 & 46^2 \\ 5'1'^2 & 52^2 & 53^2 & 54^2 & 0 & 56^2 \\ 6'1'^2 & 62^2 & 63^2 & 64^2 & 65^2 & 0 \end{vmatrix} = 0 \quad \text{.........(10).}$$

The locus is therefore a *cyclide*, and it is of the *third order;* for the

coefficient of the terms of the fourth order, viz., of $(x^2+y^2+z^2)^2$, is the determinant which, by (3), is identically zero. And, as for quadrics in § 4, the point 1 with which the correspondence is determined must also lie on a related cyclide of the third order, whose equation is

$$\begin{vmatrix} 0 & 12^2 & 13^2 & 14^2 & 15^2 & 16^2 \\ 21^2 & 0 & 2'3'^2 & 2'4'^2 & 2'5'^2 & 2'6'^2 \\ 31^2 & 3'2'^2 & 0 & 3'4'^2 & 3'5'^2 & 3'6'^2 \\ 41^2 & 4'2'^2 & 4'3'^2 & 0 & 4'5'^2 & 4'6'^2 \\ 51^2 & 5'2'^2 & 5'3'^2 & 5'4'^2 & 0 & 5'6'^2 \\ 61^2 & 6'2'^2 & 6'3'^2 & 6'4'^2 & 6'5'^2 & 0 \end{vmatrix} = 0 \quad \text{.........(11).}$$

We may say that the distances of any point P' on the first cyclide from $2'$, $3'$, $4'$, $5'$, $6'$ are to the distances of the corresponding point P on the second cyclide from 2, 3, 4, 5, 6, as $\phi(P')$ to $\phi(P)$, where $\phi(P)$ is a function of the position of P.

Now, if we take *any* five points on the first cyclide, and their related points on the other one, and find the corresponding loci for them, we arrive at a second pair of cyclides of the third order, which pass, by hypothesis, through $2'$, $3'$, $4'$, $5'$, $6'$ and 2, 3, 4, 5, 6 respectively. We thus obtain two pairs of cyclides, such that, if P, Q are points on the first pair (one on each), and P', Q' the corresponding points on the second pair,

$$\frac{PQ}{\phi(P)\,\phi(Q)} = \frac{P'Q'}{\phi(P')\,\phi(Q')} \quad \text{.....................(12).}$$

This result points to a further generalization. Instead of taking distances from corresponding fixed points proportional, we may take each distance in the first diagram proportional to a definite multiple of the corresponding distance in the second diagram. This will introduce constant multipliers into the first row and column of the determinants of (10), (11). The loci will still be cyclides, but of the fourth order, and the same reasoning as above shows that the relation (12) also holds for a system of this kind.

7. Now, considering the two pairs of cyclides, we can transfer the points 2, 3, 4, 5, 6, taken as a rigid system, into such position that the distances of any one of them from $2'$, $3'$, $4'$, $5'$, $6'$ are respectively proportional to the corresponding multiples of the distances of the related points from 2, 3, 4, 5, 6; for proportionality of distances from five points involves the same number of conditions as equality of distances from four points, and these conditions can therefore be satisfied by means of six degrees of freedom. It follows then, as for quadrics,

in § 4, that the two pairs of cyclides coincide; and we have this generalization of Ivory's theorem, that if P, Q be points on one, and P', Q' the corresponding points on the other cyclide,

$$\frac{PQ'}{\phi(P)\phi(Q')} = \frac{P'Q}{\phi(P')\phi(Q)}.$$

[7*a*. Starting from a given cyclide, it is evidently possible to form a whole series of related cyclides which shall possess this property *with respect to the former;* and we can draw surfaces of the series which shall be very close and consecutive to the original one, for, when the two surfaces coincide, the relation becomes an identity. Now take points 2, 3, 4, 5 ... on one surface indefinitely near one another, and their corresponding points 2′, 3′, 4′, 5′ ... on the consecutive surface: for our purpose we may suppose them to lie in two corresponding tangent planes, and we may suppose these planes parallel, for we thereby neglect only small quantities of the second order. For these points, the denominators in the relation (12) are equal to the first order of small quantities, and therefore the diagonal distances between pairs of corresponding points are equal, *i.e.*, $23' = 2'3$, and so on, so that the question is reduced to Ivory's case. Further, these two systems of points are homographically situated in the two parallel tangent planes. Now it is clear that, under these circumstances, the distance-equalities can only be satisfied if the lines joining corresponding points are normal to the planes, and therefore perpendicular to the surfaces.

No direct method of showing that these cyclides are related to one another as well as to the original cyclide presents itself; but it will be seen in other ways that this is true. Assuming it for the present, we can take a further step. For corresponding points lie on the curves, which, as has been seen, are the orthogonal trajectories of the system of surfaces. But these curves themselves are loci for which the relation (12) is true, for that relation may be interpreted to mean either that P and P', Q and Q' correspond, or else that P and Q, P' and Q' correspond. Now, as we have seen that all such loci are included in a system of surfaces, viz., cyclides, it follows that this congruence of orthogonal trajectories will make up a system of cyclides, which are normal to the former system. And there must clearly also be a third such system on which the curves lie; this follows by considerations of symmetry, or by considering as above the orthogonal trajectories of the second system of surfaces. The three systems of cyclides intersect everywhere at right angles, and therefore along lines of curvature on each; which is known to be a property of confocal surfaces.

The truth of the relation (12) for a system of *confocal* cyclides might also be inferred from the examples in the following sections (see § 10), but I find that Darboux has already deduced this very result from the analytical theory of a confocal system, as the analogue of Ivory's theorem. (*Sur une Classe remarquable de Courbes et de Surfaces*, Note XVI.)

If now we show that through the points 2, 3, 4, 5 and 2′, 3′, 4′, 5′ respectively there can always be drawn two quadrics which, when properly placed, are confocal with these points in correspondence, and that through the points 2, 3, 4, 5, 6 and 2′, 3′, 4′, 5′, 6′ respectively, there can always be drawn cyclides which, when properly placed, are confocal with these points in correspondence, it will follow that all the relations here investigated are *confined to* these classes of surfaces. This can be proved by counting the disposable constants.

Each quadric can satisfy 9 conditions, while the conditions of confocality absorb 2, the conditions that four points are on one quadric absorb 4 more, and the conditions that the other four points are in corresponding positions absorb 3×4 more,—making up, in all, the 18 conditions. The problem is therefore *definite* for quadrics.

Each cyclide can satisfy 13 conditions, that being the number of constants in its equation; while the condition of confocality absorbs 4, the conditions that five points lie on one cyclide absorb 5 more, and the conditions that the other five points are in the corresponding positions absorb 3×5 more—making 24 in all. There are therefore 2 of the 26 that remain arbitrary. This is in agreement with § 6; for cubic cyclides the problem would be definite, but in quartics there are additional modes of freedom.]

8. It is interesting to notice the very compact analytical solution of the problem, to draw quadrics through these two sets of four points respectively which, when properly placed, shall be confocal with these points in correspondence, that is contained in the equations (5) and (6).

When in the general case the points 6 and 6′ are both in the plane at infinity, so that they are equidistant from all other points not at infinity, we obtain a particular class of loci, viz., those for which, with four fixed reference-points, the one set of distances are *equal* respectively to definite multiples of the other set. The loci are quartic cyclides, and their equations might also be obtained by the method of § 4. If in the cubic cyclides represented by (10) and (11) we make this supposition, the surfaces reduce to confocal *quadrics;* but this merely shows that corresponding points at an infinite distance on confocal cubic cyclides must be regarded as at different infinities, and therefore not equidistant from finite points.

9. The results here investigated for confocal cyclides are of course true for the special case *in plano* of confocal bicircular quartics. For this case a direct verification is not difficult.

It is well known that a system of confocal conics can be represented by the equation in complex variables

$$x+\iota y = \sin^2(\phi+\iota\psi),$$

and Greenhill has shown that a system of confocal Cartesians may be represented by the equation

$$x+\iota y = \mathrm{sn}^2(\phi+\iota\psi),$$

where ϕ, ψ are the parameters of the two sets of mutually orthogonal curves.

Denoting the distance of the point ϕ, ψ from the point ϕ', ψ' by $\phi\psi\,.\,\phi'\psi'$, we have to show that the above relation subsists between $\phi\psi\,.\,\phi'\psi'$ and $\phi\psi'\,.\,\phi'\psi$. Now, if x, y and x', y' are the coordinates of ϕ, ψ and ϕ', ψ', we have

$$x-x'+\iota(y-y') = \mathrm{sn}^2(\phi+\iota\psi)-\mathrm{sn}^2(\phi'+\iota\psi'),$$

and, changing the sign of ι,

$$x-x'-\iota(y-y') = \mathrm{sn}^2(\phi-\iota\psi)-\mathrm{sn}^2(\phi'-\iota\psi').$$

The product of these expressions is the square of $\phi\psi\,.\,\phi'\psi'$.

Now let

$$\begin{aligned}\Theta &\equiv \{\mathrm{sn}(\phi+\iota\psi)-\mathrm{sn}(\phi'+\iota\psi')\}\{\mathrm{sn}(\phi-\iota\psi)-\mathrm{sn}(\phi'-\iota\psi')\}\\ &= \mathrm{sn}(\phi+\iota\psi)\,\mathrm{sn}(\phi-\iota\psi)-\mathrm{sn}(\phi'+\iota\psi')\,\mathrm{sn}(\phi'-\iota\psi')\\ &\quad+\tfrac{1}{2}\{\mathrm{sn}(\phi+\iota\psi)-\mathrm{sn}(\phi-\iota\psi)\}\{\mathrm{sn}(\phi'+\iota\psi')-\mathrm{sn}(\phi'-\iota\psi')\}\\ &\quad-\tfrac{1}{2}\{\mathrm{sn}(\phi+\iota\psi)+\mathrm{sn}(\phi-\iota\psi)\}\{\mathrm{sn}(\phi'+\iota\psi')+\mathrm{sn}(\phi'-\iota\psi')\}\\ &= \frac{\mathrm{sn}^2\phi-\mathrm{sn}^2\iota\psi}{1-k^2\,\mathrm{sn}^2\phi\,\mathrm{sn}^2\iota\psi}+\frac{\mathrm{sn}^2\phi'-\mathrm{sn}^2\iota\psi'}{1-k^2\,\mathrm{sn}^2\phi'\,\mathrm{sn}^2\iota\psi'}\\ &\quad+\frac{\mathrm{sn}\,\phi\,\mathrm{cn}\,\iota\psi\,\mathrm{dn}\,\iota\psi\,.\,\mathrm{sn}\,\phi'\,\mathrm{cn}\,\iota\psi'\,\mathrm{dn}\,\iota\psi'+\mathrm{cn}\,\phi\,\mathrm{sn}\,\iota\psi\,\mathrm{dn}\,\phi\,.\,\mathrm{cn}\,\phi'\,\mathrm{sn}\,\iota\psi'\,\mathrm{dn}\,\phi'}{(1-k^2\,\mathrm{sn}^2\phi\,\mathrm{sn}^2\iota\psi)(1-k^2\,\mathrm{sn}^2\phi'\,\mathrm{sn}^2\iota\psi')}.\end{aligned}$$

Hence we easily find

$$\begin{aligned}&(1-k^2\,\mathrm{sn}^2\phi\,\mathrm{sn}^2\iota\psi)(1-k^2\,\mathrm{sn}^2\phi'\,\mathrm{sn}^2\iota\psi')\,\Theta\\ &= (\mathrm{sn}^2\phi+\mathrm{sn}^2\phi')(1+k^2\,\mathrm{sn}^2\iota\psi\,\mathrm{sn}^2\iota\psi')-(\mathrm{sn}^2\iota\psi+\mathrm{sn}^2\iota\psi')(1-k^2\,\mathrm{sn}^2\phi\,\mathrm{sn}^2\phi')\\ &\quad+\mathrm{sn}\,\phi\,\mathrm{cn}\,\iota\psi\,\mathrm{dn}\,\iota\psi\,.\,\mathrm{sn}\,\phi'\,\mathrm{cn}\,\iota\psi'\,\mathrm{dn}\,\iota\psi'+\mathrm{cn}\,\phi\,\mathrm{sn}\,\iota\psi\,\mathrm{dn}\,\phi\,.\,\mathrm{cn}\,\phi'\,\mathrm{sn}\,\iota\psi'\,\mathrm{dn}\,\phi',\end{aligned}$$

which is not altered in value by permuting the accents on ϕ, ψ, ϕ', ψ'.

Changing the signs of ϕ', ψ', we obtain a similar expression for Θ', where also

$$\Theta\Theta' = (\phi\psi \, . \, \phi'\psi')^2.$$

It follows that

$$(1-k^2 \operatorname{sn}^2\phi \operatorname{sn}^2\iota\psi)(1-k^2 \operatorname{sn}^2\phi' \operatorname{sn}^2\iota\psi')(\phi\psi \, . \, \phi'\psi')$$
$$= (1-k^2 \operatorname{sn}^2\phi \operatorname{sn}^2\iota\psi')(1-k^2 \operatorname{sn}^2\phi' \operatorname{sn}^2\iota\psi)(\phi\psi' . \, \phi'\psi),$$

which is the verification sought.

The relation is unaltered by inversion, and by inverting confocal Cartesians we obtain general confocal bicircular quartics.

[9a. If ϕ, ψ are the parameters of two sets of curves, we can express the coordinates of any point in their plane in the form

$$x = \theta_1(\phi, \psi), \quad y = \theta_2(\phi, \psi),$$

where θ_1, θ_2 are functional symbols.

Now we have shown that Ivory's theorem *in plano* holds only for confocal conics, therefore the functional equation

$$\{\theta_1(\phi, \psi) - \theta_1(\phi', \psi')\}^2 + \{\theta_2(\phi, \psi) - \theta_2(\phi', \psi')\}^2$$
$$= \text{an even function of } \phi - \phi', \; \psi - \psi',$$

has a solution, which is *unique*, and represents a system of confocal conics, and can therefore be put into the form

$$\theta_1(\phi, \psi) + \iota\theta_2(\phi, \psi) + A = C \sin^2(\phi + \iota\psi),$$

where A, C are any constants, real or imaginary.

Similar analytical statements apply in the other cases.]

10. The inverses with respect to any point of a system of confocal quadrics form a special system of confocal cyclides for which the relation (12) is clearly true. Also, if the pair of cyclides of that theorem are symmetrical with respect to a plane (*i.e.*, if they have focal curves in that plane), their sections by the plane are bicircular quartics for which the theorem holds, and which must therefore, as we have seen, be confocal. These considerations [see § 7a] point to the conclusion that all the pairs of cyclides which satisfy the relation are confocal.

Assuming their confocality, and applying the relation to two focal curves, which are limiting forms of a confocal system, we arrive, by the same method as in § 1 for conics, at the result that the locus of the foci in space of a plane or spherical bicircular quartic curve is one

of the other focal curves of the system of confocal cyclides of which the original is a focal curve. This theorem has been given by Darboux. (*Sur une Classe remarquable de Courbes et de Surfaces algébraïques*, Paris, 1873, p. 44.)

March 12*th*, 1885.

J. W. L. GLAISHER, Esq., F.R.S., President, in the Chair.

Messrs. Philip Magnus, B.A., B.Sc., Director and Secretary of the City and Guilds of London Institute for the Advancement of Technical Education, and R. Lachlan, B.A., Scholar of Trinity College, Cambridge, Royal Naval College, Greenwich, were elected members.

Mr. J. J. Walker, F.R.S., made a second communication On a Method in the Analysis of Plane Curves.

Mrs. Bryant, D.Sc., read a paper On the Geometrical Form of perfectly regular Cell-structure. The President and the Treasurer made some interesting remarks in connection with the subject.

Prof. Sylvester, F.R.S., spoke On the constant Quadratic Function of the Inverse Coordinates of $n+1$ points in space of n dimensions. Prof. Cayley, F.R.S., made some remarks on this last communication.

Mr. Tucker formally communicated papers by Prof. Karl Pearson, On the Flexure of Beams; by Rev. T. C. Simmons, Two Elementary Proofs of the Contact of the Nine-point Circle of a Plane Triangle with the In- and Ex-circles, together with a Property of the Common Tangents; and Mr. Tucker mentioned that he had also obtained two elementary proofs of the first part of Mr. Simmons' paper.

The following presents were received:—

"Proceedings of the Royal Society," Vol. XXXVIII., No. 235.

"An Elementary Treatise on Dynamics," by Prof. B. Williamson and F. A. Tarleton, 8vo; London, 1885. (From the authors.)

"Mathematical Questions, with their Solutions, from the 'Educational Times,'" Vol. XLII., 8vo; London, 1885.

"Educational Times," for March.

"Bulletin des Sciences Mathématiques," Feb. 1885, and Index for 1884.

"Jahrbuch über die Fortschritte der Mathematik," xiv., 2; Jahrgang 1882; Berlin, 1885.

"Bulletin de la Société Mathématique de France," T. xii., No. 6; T. xiii., No. 1.

"Beiblätter zu den Annalen der Physik und Chemie," B. ix., St. 2; Leipzig, 1885.

"Mittheilungen der Mathematischen Gesellschaft in Hamburg," No. 5; Leipzig, 1885.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. v., No. 6; Coimbra, 1884.

"Atti della R. Accademia dei Lincei," Vol. I., F. 4, 5, and 6; Roma, 1885.

A number of pamphlets by Prof. L. Kronecker:—

Ten papers from Nos. (xx., xxi.), (xxiii., xxiv.), (xxix., xlvi.), (xxx.), (xxxvii.), (xxxviii.) of the "Abhandlungen" of the "Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin."

"Ueber bilineare Formen mit vier Variabeln," from "Crelle," Bd. xcvi., Heft 4, Bd. xcvii., Heft 2; Berlin, 1884.

"Coordonnées parallèles et axiales: Méthode de transformation géométrique et procédé nouveau de calcul graphique déduits de la considération des coordonnées parallèles," par Maurice D'Ocagne; Paris, 1885: from the Author.

Note on Certain Elementary Geometrical Notions and Determinations. By J. J. Sylvester, Savilian Professor of Geometry in the University of Oxford.

[*Read March* 12*th*, 1885.]

A curve, as every one knows, may be regarded as a *locus* of points or as an *assembly* of directions, every point being common to two consecutive directions of the assembly, and every direction to two consecutive points of the locus; the locus is called the *envelop* of the assembly (that is part of the accepted language of geometry), and, conversely, the assembly may be called the *environment* of the locus. So we may regard a surface as an assembly of tangent planes or as a locus of points standing to each other in the relation of envelop and environment, and extend these definitions to space of any number of dimensions.

By a *plasm*, waiting a better word, we may understand a figure

analogous to a point-pair in a line, a triangle in a plane, a pyramid in space, etc.; and an n-gonal plasm or n-gon will signify a plasm having n vertices and n faces themselves $(n-1)$-gons.

It is easy and desirable to find the general value of the content of a regular n-gon, say $abcde$, all whose edges we may call unity.

If $$b\beta = \tfrac{1}{2}ab, \quad c\gamma = \tfrac{2}{3}c\beta, \quad d\delta = \tfrac{3}{4}d\gamma \;\ldots,$$

it is easily seen by an elementary process of integration that $\beta, \gamma, \delta \ldots$ are the centres of figure to the successive plasms ab, abc, $abcd$, ..., and,

making $$ba = p_1, \quad c\beta = p_2, \quad d\gamma = p_3 \ldots,$$

each term in $p_1, p_2, p_3 \ldots$ will be perpendicular to the one which precedes it, so that, if V_n is the content of the plasm,

$$(1, 2, 3 \ldots n)^2\, V_n = p_1 p_2 \ldots p_n.$$

Moreover, we shall have

$$p_n^2 = 1 - \left(\frac{n-1}{n}\right)^2 p_{n-1}^2,$$

of which the general integral is

$$p_n^2 = \frac{n+1}{2 \, . \, n} + C\,(-)^n \frac{1}{n^2};$$

in the present case, since $p_1 = 1$, $C = 0$, so that

$$V_n^2 = \frac{n+1}{(1 \, . \, 2 \ldots n)^2\, 2^n}.$$

If a, b, c be the angles of a fixed triangle, and A, B, C are proportional to the distances of a variable line from a, b, c respectively, we may denote the line by $A : B : C$; as regards a variable point, it will presently be seen to be advantageous to denote its proportional coordinates, not, as is rather more usually done, by equimultiples of its distances from the three sides, but as equimultiples of these distances multiplied by the sides of the triangle from which they are measured*; so that, calling these coordinates a, b, c, the image† of the line at infinity becomes $a+b+c$.

Consider now the universal mixed concomitant (which it will be

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* Or rather divided by the distances of these sides from the opposite angles of the fundamental triangle, whose coordinates thus become 1, 0, 0, 0, 1, 0, 0, 0, 1.

† If F = 0 is the equation to any *locus* or *assembly*, I call F the *image*, and such locus or assembly the *object*; to a given image responds in general an absolutely definite object, but, when the object is given, the image is only determined to a *constant* factor *près*.

convenient to call a *mutuant*) $Aa+Bb+Cc$ (where a, b, c, A, B, C are used in lieu of the more usual letters $x, y, z, \xi, \eta, \zeta$); it will readily be seen that, when a, b, c vary, and A, B, C are fixed, the mutuant images the line $A : B : C$, and that, when A, B, C vary and a, b, c are fixed, the mutuant images the *radiant* point $a : b : c$; that is to say, $Aa+Bb+Cc \mathrel{\overset{}{=}} 0$ is true for every point in the point-containing line $A : B : C$ in the one case, and to every line through the *radiant point* $a : b : c$ in the other.

Supposing, then, that the two kinds of coordinates are chosen in this manner, we see (what would not be the case if the simple distances were taken) that a form F and its "polar-reciprocal" ϕ image the self-same curve referred to the self-same fundamental triangle.

These consequences would moreover continue to subsist if, calling the distances of a line from the vertices P, Q, R, and of a point from the sides p, q, r, we took $\Lambda P : \mathrm{M}Q : \mathrm{N}R, \lambda p : \mu q : \nu r$ for the two sets of coordinates, provided only that $\lambda\Lambda F = \mu \mathrm{M} G = \nu \mathrm{N} H$; F, G, H being the distances of the sides from the vertices of the fundamental triangle, in which case the line at infinity would no longer be imaged by $a+b+c$. I shall, however, adhere in what follows to the convention above laid down. I need hardly add that in like manner, in space taking $A : B : C : D$ (the distances of a plane from the vertices of a fundamental pyramid) as the coordinate representation of such plane, and $a : b : c : d$ (the contents of the volumes which any variable point makes with the respective faces) as the coordinate-representation of such point, the mutuant $aA+bB+cC+dD$ will be the image of the radiant point $a : b : c : d$ when the capital letters are the variables, and of the plane $A : B : C : D$ when the small letters are the variables, meaning of course that $Aa+Bb+Cc+Dd = 0$ will be true of every point in the plane $A : B : C : D$ and of every plane through the point $a : b : c : d$, and, as before, F and ϕ polar-reciprocals to each other will image the self-same surface (referred to the self-same fundamental pyramid) viewed as a locus or envelop on the one hand, as an assembly or environment on the other.

If a, b, c, d be used to signify the actual as distinguished from the proportional coordinates of a point, a linear function of these is constant, whereas it is a quadratic function of $A, B, C, D \ldots$, when used to signify the actual distances of a variable line, plane, &c. from the vertices of the fundamental plasm which is constant; and it is the principal object of this note to determine the form of this quadratic function, which, as Prof. Cayley was the first to show, may be expressed by the determinant to a matrix standing in close relation to the well-known "invertebrate symmetrical matrix," the determinant to which represents a numerical multiple of any plasm

in terms of its edges, as *ex. gr.*:

$$\begin{vmatrix} . & ab & ac & ad & 1 \\ ba & . & bc & bd & 1 \\ ca & cb & . & cd & 1 \\ da & db & dc & . & 1 \\ 1 & 1 & 1 & 1 & . \end{vmatrix}.$$

where ab, ac, bc ... are used for brevity to signify the measure of absolute distance between a, b, a, c, b, c ..., *i.e.*, stand for what in ordinary notation would be denoted by $(ab)^2$, $(ac)^2$, $(bc)^2$, ... This may be quoted as the mutual-distance matrix; its determinant, besides representing a numerical multiplier of the squared content of the pyramid when equated to zero, expresses the conditions of the four points a, b, c, d lying in a plane, the former property being a consequence immediately deducible by *strict* algebraical reasoning from the latter.

That this determinant does image the condition of the plasm to which the points a, b, c, d ... are the vertices, losing one dimension of space, may be shown in a somewhat striking manner as follows. If for a moment we use x, y, z, the distances of any point in the plane of abc from bc, ca, ab as coordinates, the equation to a circle circumscribed about abc will be of the form $fyz+gzx+hxy$, and, calling the sides of the triangle a, b, c respectively, $ax+by+cz$ is constant. Hence, substituting for z its value in terms of x and y, the image of the circle may be put under a form in which fb and ga will be the coefficients of y^2 and x^2 respectively; but, since x and y are *proportional* to the Cartesian coordinates y and x respectively, the coefficients of x^2 and y^2 must be equal. Hence $f : g : h :: a : b : c$, and if now ax, by, cz, instead of x, y, z, be used as the coordinates of the variable point, the image to the circumscribing circle becomes $\Sigma \dfrac{ayz}{bc}$, or if we please Σa^2yz, *i.e.*, $\Sigma\, bc\, yz$, where bc stands as convened for $(bc)^2$.

Hence, if a, b, c, d be the vertices of a pyramid, $\Sigma\, ab\, yz$ will be the image of the circumscribing sphere, for when any coordinate t is made zero the image becomes that of a circle; and so universally for a plasm of any number of dimensions.

Consider the case of a circle, and suppose that

$$\begin{vmatrix} . & ab & ac & 1 \\ ba & . & bc & 1 \\ ca & cb & . & 1 \\ 1 & 1 & 1 & . \end{vmatrix}$$

vanishes; this means that the line $x+y+z$ touches the circle

$$ab\,xy+bc\,yz+ca\,zx.$$

But, if $x+y+z$ images the line at infinity, it must *cut* this (as it cuts any other circle) in two distinct points, viz., the so-called circular points at infinity. Hence $x+y+z$ must, when the above determinant vanishes, cease to be the line at infinity, which can only come to pass by the triangle abc losing a dimension of space, and a, b, c coming into a straight line, in which case $x+y+z=0$, instead of being true of a particular line, is true of every point in the plane.

Just in like manner, if

$$\begin{vmatrix} \cdot & ab & ac & ad & 1 \\ ba & \cdot & bc & bd & 1 \\ ca & cb & & cd & 1 \\ da & db & dc & \cdot & 1 \\ 1 & 1 & 1 & 1 & \cdot \end{vmatrix}$$

vanishes, unless $x+y+z+t$ ceases to image the plane at infinity, this plane would touch the sphere $\Sigma\,ab\,xy$, *i.e.*, would cut it in a pair of straight lines, whereas it intersects it in a circle. Consequently the plasm $abcd$ must, as before, lose one dimension, and so in general. The content of a plasm vanishes when the mutual-distance determinant does so, and the latter as well as the former may be expressed rationally in terms of ordinary Cartesian coordinates; but the expression for the content (being linear in each set of coordinates) is obviously indecomposable, and must therefore be a numerical multiple of some power of the mutual-distance determinant; a comparison of dimensions shows at once that this power is the square root.

As regards the numerical multiplier, when the plasm has all its edges equal to unity (say a triangle, for example), the mutual-distance determinant becomes

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix},$$

which is easily transformable into

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & \bar{1} & 0 & 0 \\ 1 & 0 & \bar{1} & 0 \\ 1 & 0 & 0 & \bar{1} \end{vmatrix},$$

of which the value is -3; and so in general for a regular plasm with $(n+1)$ vertices; *i.e.*, in space of n dimensions the mutual-distance determinant, say D_n, becomes $(-)^{n+1}(n+1)$, whereas the (volume)^2, say V_n^2, has been shown to be $\dfrac{n+1}{2^n(1.2\ldots n)^2}$.

Hence, universally,

$$D_n = (-)^{n+1}2^n(1.2\ldots n)^2 V_n^2.$$

It may be here noticed that, if p be the perpendicular from any vertex on an opposite face of the plasm whose content is V_{n-1}, we shall have

$$V_{n-1}p = nV_n.$$

Consequently,

$$\begin{aligned} D_{n-1}p^2 &= (-)^n 2^{n-1}(1.2\ldots\overline{n-1})^2 V_{n-1}^2 p^2 \\ &= (-)^n 2^{n-1}(1.2\ldots n)^2 V_n^2 = -\tfrac{1}{2}D_n. \end{aligned}$$

I now pass on to the leading motive of this note, viz., the determination of the connection between the coordinates $A, B, C \ldots$ drawn from $a, b, c \ldots$.

It is clear *à priori* that the form of the condition will be in all cases that a homogeneous quadratic function of the distances must be constant. Thus, *ex. gr.*, when there are four points, if A, B, C be assumed, we may describe three spheres with these quantities as radii, and the fourth point will be determined by means of one of the pairs of tangent planes drawn to them, the particular pair depending on the relative signs attributed to A, B, C. Hence, if $F(A, B, C, D) = \infty$ be the general equation, each of the quantities must enter in the second and no higher degree; moreover, since by transporting the plane from which the distances are measured parallel to itself, A, B, C, D will be all increased by the same quantity, F must express a function of their differences, and consequently, since any two distances may be interchanged, F can contain no terms of the first order in the variables, so that $F=0$ must amount to the predication of a homogeneous quadratic function of the distances being constant.

Thus, *ex. gr.*, in the case of three points, we have the well-known

equation $\quad \Sigma(ab)(A-C)(B-C) = \tfrac{1}{4}(abc)^2.$

Suppose now that A, B, C are taken in proportions consistent with

making $\quad \Sigma(ab)^2(A-C)(B-C) = 0.$

Let $\Sigma(ab)^2(A-C)(B-C) = P.Q$, where P, Q are two linear functions of A, B, C; then P, Q image two radiant points, each of

which will have the property that any of its rays is at an infinite distance from a, b, c, or at all events, if it should pass through one of them, from the other two, and it is easy to anticipate that these two points must be the circular points at infinity. That such is the fact is obvious, because (using Cartesian coordinates) the perpendicular distance from any point upon $x \pm \sqrt{-1}\,.\,y$ contains zero in its denominator; so that the two points of the absolute may be regarded as the centres of two points of rays, all of them infinitely distant from the finite region.

But these two points are the intersections of the circumscribing circle with the line at infinity, and consequently their collective equation will be found by taking the resultant of $\Sigma\, ab\, xy$, $\Sigma\, x$, $\Sigma A x$, which is well known to be the determinant of the quadratic function bordered by the coefficients of the two linear ones. Hence the constant quadratic function in A, B, C, viz. $\Sigma\, ab\,(A-B)\,(A-C)$, ought to be a numerical multiple of the determinant

$$\begin{vmatrix} . & A & B & C & . \\ A & . & ab & ac & 1 \\ B & ba & . & bc & 1 \\ C & ca & cb & . & 1 \\ . & 1 & 1 & 1 & . \end{vmatrix},$$

as is the case, the value of this determinant being

$$-2\Sigma\, ab\,(A-C)\,(B-C).$$

The same thing may be shown in a more elementary manner as follows. Combining

$$x+y+z=0, \qquad ab\,xy+bc\,yz+ca\,zx=0,$$

we have $$acx^2+(bc+ca-ab)\,xy+bcy^2=0,$$

at each point of the absolute. And, taking $x_1 y_1 z_1$, $x_2 y_2 z_2$ as the coordinates at these two points, it follows that

$$x_1x_2 : y_1y_2 : z_1z_2 : x_1y_2+x_2y_1 : y_1z_2+y_2z_1 : z_1x_2+z_2x_1$$
$$:: bc : ca : ab : -bc-ca+ab : -ca-ab+bc : -ab-bc+ca.$$

And, as the two points will be imaged by

$$x_1A+y_1B+z_1C, \qquad x_2A+y_2B+z_2C,$$

respectively, it follows that their collective image will be

$$\Sigma \{bcA^2 + (bc - ab - ac) BC\},$$

which is easily seen to be identical with

$$\Sigma bc (A - B)(A - C).$$

The universal algebraical theorem upon which the first method of proof depends is the well-known one that, if Q is a quadratic function and $L_1, L_2, \ldots L_i$ i linear functions of j variables, and if Q' (where j is not less than $i+1$) is what Q becomes when i of its variables are expressed in terms of the rest, then the necessary and sufficient condition of the discriminant of *every* such Q' vanishing is that the determinant to Q bordered by the coefficients of the i linear functions shall vanish. When j is equal to $i+1$, the theorem shows that the resultant of the quadratic and its i attendant linear functions will be the bordered determinant in question. In the above example we had $j = 3$, $i = 2$.

Let us now proceed to apply a similar principle to the case of four points a, b, c, d in space.

If we take the case $x^2+y^2+z^2+t^2 = 0$, any tangent plane to it at

x', y', z', t' will be $\qquad x'x+y'y+z'z+t't,$

and, as $\qquad x'^2+y'^2+z'^2+t'^2 = 0,$

it follows that every tangent plane will be at infinite distance from any point external to it; and, as this is true wherever the centre of the cone be placed, and all the cones so obtained have the "circle at infinity" in common,—it follows that every tangent plane to the circle at infinity is infinitely distant from any external point in the finite region,—the infinitely-infinite system of planes thus obtained one may regard, if one pleases, as consisting of sheaves of planes whose axes form the environment to the circle at infinity, and will be the correlative to the infinitely-infinite system of points in the plane at infinity, which are infinitely distant from all external planes in the finite region. We see, then, that the coordinates to each such plane must satisfy the condition that, on making $\Sigma x = 0$ and $\Sigma Ax = 0$, and expressing any two of the variables x, y, z, t in terms of the two others, the discriminant of the form then assumed by $\Sigma abxy$ must vanish, and consequently, as before, the mutual-distance determinant to the points a, b, c, d, bordered with a row and column of units and a row and column consisting of the letters A, B, C, D, will represent to a numerical factor *près* the constant quadratic function of distances,

i.e., this function will be

$$\begin{vmatrix} & A & B & C & D & . \\ A & . & ab & ac & ad & 1 \\ B & ba & . & bc & bd & 1 \\ C & ca & cb & . & cd & 1 \\ D & da & db & dc & . & 1 \\ . & 1 & 1 & 1 & 1 & . \end{vmatrix},$$

and obviously a similar algebraical conclusion will continue to apply, whatever may be the number of points n in a space of $n-1$ dimensions.

As regards the value of the constant, in any case, that may be obtained by taking a face of the plasm as the *term* (line, plane, etc.) from which the distances A, B, C ..., are measured; *i.e.*, we may make $B=0$, $C=0$, $D=0$ ..., provided we make A equal to the perpendicular from a on the opposite face. The value of the bordered determinants then becomes the *negative* of the squared perpendicular from a on bcd ... multiplied by the mutual-distance determinant to bcd ...; *i.e.*, by virtue of what has previously been shown, will be $\frac{1}{2}$ of the mutual-distance determinant of $abcd$ ...

Hence the complete relation between A, B, C, D may be exhibited by making

$$\begin{vmatrix} -\frac{1}{2} & A & B & C & D & \\ A & . & ab & ac & ad & 1 \\ B & ba & . & bc & bd & 1 \\ C & ca & cb & . & cd & 1 \\ D & da & db & dc & . & 1 \\ & 1 & 1 & 1 & 1 & \end{vmatrix} = 0,$$

and similarly for any number of points.

Professor Cayley has obtained the same result by a more direct but not more instructive process, as follows. Taking, by way of example, three points, $A+k$, $B+k$, $C+k$, (where k is infinite,) may be regarded as the distances of a, b, c from a fourth point at an infinite distance, and accordingly we may write

$$\begin{vmatrix} . & ab & ac & (A+k)^2 & 1 \\ ba & . & bc & (B+k)^2 & 1 \\ ca & cb & . & (C+k)^2 & 1 \\ (A+k)^2 & (B+k)^2 & (C+k)^2 & . & 1 \\ 1 & 1 & 1 & 1 & . \end{vmatrix} = 0.$$

For the *gnomon* bordering the square formed by the small letters and dots, we may substitute

$$\begin{vmatrix} & & & 2kA+A^2 & 1 \\ & & & 2kB+B^2 & 1 \\ & & & 2kC+C^2 & 1 \\ 2kA+A^2 & 2kB+B^2 & 2kC+C^2 & -2k^2 & 1 \\ 1 & 1 & 1 & 1 & \end{vmatrix},$$

without altering the value of the determinant, which therefore, remembering that k is infinite, is in a ratio of equality to $(2k)^2$ multiplied into the determinant

$$\begin{vmatrix} . & ab & ac & A & 1 \\ ba & . & bc & B & 1 \\ ca & cb & . & C & 1 \\ A & B & C & -\frac{1}{2} & . \\ 1 & 1 & 1 & . & . \end{vmatrix}.$$

This last determinant therefore must vanish, agreeing with what has been shown above by a more purely geometrical method.* I will now proceed to develop this determinant deprived of its constant term, expressing it as a function of the differences of the capital letters.

It is obvious that it may be expressed as a sum of terms of which each variable part will be of one or the other of these three forms $(A-B)^2$, $(A-B)(A-C)$, $(A-B)(C-D)$; and accordingly we may distribute the totality of the terms of the constant function of difference into three families depending on the form of the variable argument.

In general, if we consider any *invertebrate* symmetrical determinant

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* As a corollary, we may infer, from the vanishing of this determinant, that, using the notation previously employed, $\frac{D_n}{V_n^2} = -\frac{1}{2}n^2\frac{D_{n-1}}{V_{n-1}^2}$,

and consequently that $D_n = -(2)^n(1.2\ldots n)^2 V_n^2$,

and that thus the content of a regular plasm with unit edges and $(n+1)$ vertices is

$$\frac{n+1}{2^n(1.2\ldots n)^2}, \text{ viz., } \frac{3}{16}, \frac{1}{72}, \frac{5}{9.2^{10}}\ldots$$

for triangle, pyramid, plu-pyramid, etc.

expressed by the *umbral* notation

$$\begin{vmatrix} aa & ab & ac & \dots & al \\ ba & bb & bc & \dots & bl \\ \dots & \dots & \dots & & \dots \\ la & lb & lc & \dots & ll \end{vmatrix},$$

where $aa = bb = cc = ll \dots = 0$ and $pq = qp$, we have this simple rule of proceeding:

Divide the letters $a \dots l$ in every possible manner into cyclical sets, each set containing at least two letters.

Any cycle $a_1a_2 \dots a_i$ is to be interpreted as meaning

$$a_1a_2 \,.\, a_2a_3 \dots a_{i-1}a_i \,.\, a_1a_i,$$

which, by virtue of the supposed condition $ab = ba$, will be the same in whichever direction the cycle is read, the effect of the inversion of the cycle being merely to give the same product over again, written under the form $a_1a_i \,.\, a_2a_1 \dots a_1a_{i-1}$.

The cycle of two letters a_1a_2 must be interpreted to mean $(a_1a_2)^2$. If now $C_1C_2 \dots C_i$ are cycles of two letters each, and $\chi_1\chi_2 \dots \chi_j$ cycles of three or more letters, the total value of the determinant will be

$$\Sigma\,(-)^{n+i+j}\, 2^j C_1C_2 \dots C_i\chi_1\chi_2 \dots \chi_j.$$

If, the principal diagonal terms remaining zero, the other terms were general, then the expression of the value of the determinant, calling the cycles $C_1C_2 \dots C_\nu$, and making no distinction between the case of their being binary or super-binary, would be $\Sigma\,(-)^{n+\nu}C_1C_2 \dots C_\nu$; only it would have to be understood that each cycle of two letters, as (ab), would mean $(ab)^2$, but a cycle of three or more letters, as (abc), would mean $ab \,.\, bc \,.\, ca + ac \,.\, cb \,.\, ba$.

This being premised, it is easy to deduce the following rule for the determination of the *three* different families of terms belonging to the constant determinant of distances, which, to avoid prolixity, must be left to the reader to verify.

Family I. — Omitting any two letters, and forming all possible cyclical products with the remaining $(n-2)$ letters, if $C_1C_2 \dots C_\nu$ be any set thereof, and ν' the number of them containing more than two letters, the general term will be $\Sigma\Sigma\,(-)^{n+\nu}2^{\nu'}\,C_1 \,.\, C_2 \dots C_\nu\,(A-B)^2$, a, b being the two letters which do not occur in the cycles $C_1C_2 \dots C_\nu$.

Family II.—Omitting any one letter, and forming with the remaining $n-1$ letters, in every possible way, *a chain* χ containing two or more

letters, and cycles $C_1 C_2 \dots C_\nu$, then, supposing the chain to be $bcd \dots kl$, and understanding by (χ) the product $bc \,.\, cd \dots kl$, the general term will be $\Sigma\Sigma\,(-)^{n+\nu}\,2^{\nu'+1}\,C_1 C_2 \dots C_\nu\,(\chi)\,(A-B)\,(A-L)$, a being the letter which does not appear in the chain or any of the cycles, and ν' meaning as before the number of the cycles which contain at least three elements.

FAMILY III.—Form all the letters in every possible way into two *chains* (each containing two or more letters) χ, χ', and into cycles C_1, $C_2, \dots C_\nu$; then, supposing the initial and final letters of χ to be a, h, and of χ' to be k, l, the general term of this family will be

$$2\Sigma\,(-)^{n+\nu}2^{\nu'+1}C_1C_2 \dots C_\nu\,(\chi)\,(\chi')\,\{(A-K)\,(H-L)+(A-L)(H-K)\}.$$

I subjoin in the following table the *types* of the coefficients of the several families for all the values of n from 2 up to 7; the vacant cycle ( ) of course means unity, and a cycle (ab) means $(ab)^2$; *i.e.*, the *fourth* power of the length ab.

Every cycle enclosed in a parenthesis of three or more letters, will be understood to be affected with a coefficient 2, and for greater brevity the variable part of each term is *left to be supplied.* A round parenthesis indicates a cycle, a square parenthesis a chain.

Number of Letters.	Types.	Name of Family.
2	( )	1st
3	(bc)	2nd
4	$-(cd)$	1st
,,	$2\,[bcd]$	2nd
,,	$2\,[ab]\,.\,[cd]$	3rd
5	(cde)	1st
,,	$-2\,[bcde] : 2\,(bc)\,[de]$	2nd
,,	$-2\,[ab]\,[cde]$	3rd
6	$-(cdef) : (cd)\,(ef)$	1st
,,	$-2\,(bcd)\,[ef] : -2\,(bc)\,[def] : -2\,[bcdef]$	2nd
,,	$-2\,(ab)\,[cd]\,[ef] : [abc]\,[def] : [ab]\,[cdef]$	3rd
7	$(cdefg)-(cd)\,(efg)$	1st
,,	$2\,(bcde)[fg] : 2(bcd)[efg] : -2(bc)(de)[fg]\;2(bc)[defg] : -2[bcdefg]$	2nd
,,	$2\,(abc)[de][fg] : 2\,(ab)[cd][efg] : -2\,[abc][defg] : -2\,[abcdefg]$	3rd.

Thus, *ex. gr.*, the constant function of distances for three points in a plane is $2\Sigma\, bc\,(A-B)\,(A-C)$; for four points in space is

$$-\Sigma\, cd\,(A-B)^2+2\Sigma\, bc\,.\,cd\,(A-B)\,(A-D)$$
$$+2\Sigma ab\,.\,cd\,(\overline{A-C}\;\overline{B-D}+\overline{A-D}\;\overline{B-C});$$

for five points in hyper-space is

$$2\Sigma\,(cd\,.\,de\,.\,ec)\,(A-B)^2-2\Sigma\,(bc\,.\,cd\,.\,de)\,(A-B)\,(A-E)$$
$$+2\,(bc)^2\,(de)\,(A-D)\,(B-E)$$
$$-2\Sigma\, ab\,.\,cd\,.\,de\,.\,ec\,\big((A-C)(B-E)+(A-E)(B-C)\big).$$

The part of the constant function of distances for seven points belonging to the 2nd family of terms will be

$$4\Sigma\, bc\,.\,cd\,.\,de\,.\,eb\,.fg\,(A-B)(A-E)+4\Sigma bc.cd\,.\,db\,.ef.\,fg\,(A-E)(A-G)$$
$$-2\,(bc)^2\,(de)^2fg\,(A-F)(A-G)+2\,(bc)^2\,(de\,.\,ef\,.fg)(A-D)(A-G)$$
$$-2\,bc\,.\,cd\,.\,de\,.\,ef\,.fg\,(A-B)\,(A-G).$$

The number of types in each family for n points is easily expressible by a generating function.

Obviously in the 1st family this number is the number of ways of resolving n into parts none less than 2; *i.e.*, it is the coefficient of

x^{n-2} in $$\frac{1}{1-x^2\,.\,1-x^3\,.\,1-x^4\ldots}.$$

In the 2nd family, it is the sum of the number of ways of decomposing $n-3$, $n-4$, ... into parts none less than 2; *i.e.*, it is the coefficient of x^{n-3} in

$$\frac{1+x+x^2+\ldots}{(1-x^2)(1-x^3)\ldots},\;\; i.e. \text{ in } \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots}.$$

In the 3rd family, if the number of ways of dividing r into two parts, neither of them less than 2, is called (r), and of dividing $(n-r)$ into any number of parts, none less than 2, is called $[n-r]$, the number of types is $\Sigma\,(r)\,[n-r]$; *i.e.*, it is the coefficient of x^{n-4} in

$$\frac{1+x+2x^2+2x^3+3x^4+3x^5+\ldots}{(1-x^2)\,(1-x^3)\,(1-x^4)\ldots},\;\; i.e. \text{ in } \frac{1}{(1-x)(1-x^2)^2(1-x^3)(1-x^4)\ldots}.$$

Hence the total number of types in all three families combined will be the coefficient of x^{n-2} in

$$\frac{(1-x)(1-x^2)+x\,(1-x^2)+x^2}{1-x\,.\,1-x^2\,.\,1-x^3\ldots},\;\; i.e. \text{ in } \frac{1}{1-x\,.\,(1-x^2)^2.\,1-x^3.\,1-x^4\ldots}.$$

Consequently, the indefinite partitions of 0, 1, 2, 3, 4, 5, 6, 7, ... being 1, 1, 2, 3, 5, 7, 11, 15, ..., the series for the type-number will be found by summing all the terms in the odd and even places successively. We thus obtain the series 1, 1, 3, 4, 8, 11, 19, 26, ... for the number of types in the constant-distance function for 2, 3, 4, 5, 6, 7, 8, 9, ... points respectively.

It may be worth while to exhibit the rule for the formation of the constant function of distances under a slightly different aspect.

As before, by the reading of any cycle, understand the product of its successive duads affected with the multiplier -1 or -2, according as the number of letters in the cycle is two or more than two.

By a *modified* reading of a cycle, understand what the reading becomes on substituting for any two duads pq, rs the product $(P-Q)(R-S)$, as for instance $(A-B)(C-D)$ in lieu of $ab \cdot cd$, $(A-B)(B-C)$ in lieu of $ab \cdot bc$, and (which can only happen in the case of a cycle of two letters), $(A-B)(B-A)$, *i.e.*, $-(A-B)^2$ in lieu of $ab \cdot ba$.

Then, to find the constant function of distances to any given set of letters, we must begin with distributing the letters in every possible way into cycles containing between them two or more letters. Each such combination of cycles we may call a distribution.

In each distribution the cycle is to be taken (each in its turn), and the *sum* of its modified readings is to be multiplied by the product of the readings of the remaining cycles, if there are any. The sum of these sums (or the single sum, if there is but one cycle) is the portion of the quadratic function sought, due to the particular distribution dealt with; and the sum of these double sums, taken for each distribution in succession, is the total value of the function, and will be equal exactly to its representative determinant when the number of letters is odd, and to the same with its sign changed when that number is even.

As an example for five letters a, b, c, d, e, there will be ten distributions of the form $(ab)(cde)$, and twelve distributions of the form $(abcde)$.

From any one of the first ten distributions, as $(ab)(cde)$, by modifying first (ab) and then (cde), we obtain

1°. $2\,(cd \cdot de \cdot ec)(A-B)(B-A)$,

2°. $2\,(ab)^2\{ce(C-D)(D-E)+dc(D-E)(E-C)+ed(E-C)(C-D)\}$.

And from a distribution of the form $(abcde)$ we obtain, by operating on consecutive duads,

$$5 \text{ terms of the form } -2\{cd \cdot de \cdot ea\,(A-B)(B-C)\},$$

and, by operating on non-consecutive duads,

$$5 \text{ terms of the form } -2\,\{bc\,.\,de\,.\,ea\,\overline{A-B}\;\overline{C-D}\}^{*}.$$

The sum of all the sums of terms due to the twenty-two distributions is the constant function of distances for the five given letters.

In the case of six letters the distributions into cycles will be of four kinds, corresponding to the partitions 6; 4, 2; 3, 3; 2, 2, 2.

The first kind will contain two types of the 3rd family and one of the 2nd family; the second kind will contain one type of each of the three families, and the third and fourth kinds single types of the 2nd and 1st families respectively, thus giving eight distinct types of terms in all, as should be the case according to the rule.

On a Method in the Analysis of Plane Curves. Part II.

By J. J. Walker. M.A., F.R.S.

[*Read March* 12*th*, 1885.]

VI. *A second form of D^2v.*

Besides the forms of developments of the different powers of the operator

$$q\,(q-1)\,\dots\,(q-p+1)\,D^p$$

$$=\left\{(m\sin C-n\sin B)\frac{d}{dx}+(n\sin A-l\sin C)\frac{d}{dy}+(l\sin B-m\sin A)\frac{d}{dz}\right\}^p,$$

used in the former part of this paper (*Proceedings*, Vol. IX., pp. 226—242), in treating the intersection of a transversal $lx+my+nz$ with the curve (of order q) $v=0$, founded on that of $q\,(q-1)\,D^2v$, there

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* It will be observed that the distribution (*acbde*) will give a term

$$-2\left\{cb\,.\,de\,.\,ea\,\overline{A-C}\;\overline{B-D}\right\},$$

in which the literal part $cb\,.\,de\,.\,ea$ is equal to the literal part $bc\,.\,de\,.\,ea$ in the term above expressed. This is how it comes to pass that the terms of the 3rd family may be grouped in pairs, as stated in the prior mode of arranging the result according to families instead of according to cycles.

exists a second system; viz., if $\Delta = x\sin A + y\sin B + z\sin C$,

$$q(q-1)\,xyzD^2v = -\Delta^2\left(l^2x\frac{d^2v}{dy\,dz} + m^2y\frac{d^2v}{dz\,dx} + n^2z\frac{d^2v}{dx\,dy}\right)$$
$$+(q-1)\,\Delta\left\{(m^2y\sin C + n^2z\sin B)\frac{dv}{dx} + \dots\right\}$$
$$-q(q-1)\,(l^2x\sin B\sin C + m^2y\sin C\sin A + n^2z\sin A\sin B)\,v.$$

To prove this, $\qquad q(q-1)\,xyzD^2v$

$$= l'\left(l'x\frac{d^2v}{dx^2} + m'x\frac{d^2v}{dx\,dy} + n'x\frac{d^2v}{dz\,dx}\right)yz$$
$$+m'\left(l'y\frac{d^2v}{dx\,dy} + \dots\right)zx + n'\left(l'z\frac{d^2v}{dz\,dx} + \dots\right)xy,$$

where l', m', n' stand for $m\sin C - n\sin B$, $n\sin A - l\sin C$, $l\sin B - m\sin A$, so that

$$m'x = nx\sin A - lx\sin C = nx\sin A + nz\sin C + my\sin C = n\Delta + l'y.$$

Similarly $n'x = -m\Delta + l'z$, $n'y = l\Delta + m'z$, $m'z = -l\Delta + n'y$.

Making these substitutions,

$$q(q-1)\,xyzD^2v = (q-1)\left(l'^2yz\frac{dv}{dx} + m'^2zx\frac{dv}{dy} + n'^2xy\frac{dv}{dz}\right)$$
$$+\Delta\left\{nz(l'y - m'x)\frac{d^2v}{dx\,dy} + my\,(n'x - l'z)\frac{d^2v}{dz\,dx} + lx\,(m'z - n'y)\frac{d^2v}{dy\,dz}\right\}.$$

Now $\qquad l'^2 = \Delta\,(m^2y\sin C + n^2z\sin B)$

$$-x\,(l^2x\sin B\sin C + m^2y\sin C\sin A + n^2z\sin A\sin B),$$

with similar values for m'^2zx, n'^2xy; and, as has been shown,

$$m'z - n'y = -l\Delta,\quad n'x - l'z = -m\Delta,\quad l'y - m'x = -n\Delta;$$

so that, finally,

$$q(q-1)\,xyz\,D^2v = -\Delta^2\left(l^2x\frac{d^2v}{dy\,dz} + m^2y\frac{d^2v}{dz\,dx} + n^2z\frac{d^2v}{dx\,dy}\right)$$
$$+(q-1)\,\Delta\left\{(m^2y\sin C + n^2z\sin B)\frac{dv}{dx} + \dots\right\}$$
$$-q(q-1)(l^2x\sin B\sin C + m^2y\sin C\sin A + n^2z\sin A\sin B)\,v,$$

or if, for shortness,

$$q(q-1)\,f = \frac{d^2v}{dy\,dz},\ \dots,\quad qv' = \frac{dv}{dx},\quad qv'' = \frac{dv}{dy},\quad qv''' = \frac{dv}{dz},$$

and α, β, γ stand for $\sin A$, $\sin B$, $\sin C$,

$$xyz\,D^2v = -\Delta^2\,(l^2xf+m^2yg+n^2zh)$$
$$+\Delta\,\{(m^2y\gamma+n^2z\beta)\,v'+\ldots\}-(l^2x\beta\gamma+m^2y\gamma\alpha+n^2z\alpha\beta)\,v\ldots(31).$$

A precisely analogous demonstration proves that, v, w being two functions of orders q, q' respectively,

$$2qq'xyzDv\times Dw = -\Delta^2\left\{l^2x\left(\frac{dv}{dy}\frac{dw}{dz}+\frac{dw}{dy}\frac{dv}{dz}\right)+m^2y\left(\frac{dv}{dz}\frac{dw}{dx}+\frac{dw}{dz}\frac{dv}{dx}\right)\right.$$
$$\left.+n^2z\left(\frac{dv}{dx}\frac{dw}{dy}+\frac{dw}{dx}\frac{dv}{dy}\right)\right\}$$
$$+\Delta\left\{(m^2y\sin C+n^2z\sin B)\left(qv\frac{dw}{dx}+q'w\frac{dv}{dx}\right)\right.$$
$$+(n^2z\sin A+l^2x\sin C)\left(qv\frac{dw}{dy}+q'w\frac{dv}{dy}\right)$$
$$\left.+(l^2x\sin B+m^2y\sin A)\left(qv\frac{dw}{dz}+q'w\frac{dv}{dz}\right)\right\}$$
$$-2qq'(l^2x\sin B\sin C+m^2y\sin C\sin A+n^2z\sin A\sin B)\,vw.$$

In particular,

$$q^2xyz\,(Dv)^2 = -\Delta^2\left\{l^2x\left(\frac{dv}{dx}\right)^2+m^2y\left(\frac{dv}{dy}\right)^2+n^2z\left(\frac{dv}{dz}\right)^2\right\}$$
$$+q\Delta\left\{(m^2y\sin C+n^2z\sin B)\frac{dv}{dx}+\ldots\right\}v$$
$$-q^2\,(l^2x\sin B\sin C+\ldots)\,v^2,$$

or
$$xyz\,(Dv)^2 = -\Delta^2\,(l^2xv'^2+\ldots)+\Delta\,\{(m^2y\gamma+n^2z\beta)\,v'+\ldots\}$$
$$-\,(l^2x\beta\gamma+\ldots)\,v^2\;\ldots\ldots\ldots\ldots\;(32).$$

As in some cases the developments previously given are more readily applicable, or give results in simpler forms, so in other cases these latter developments have the advantage; and they have the further interest of being the analogues of the only form of development which exists for the corresponding method in space of three dimensions.

VII.

The Quartic Node-Tangentials Line.

As another example of the application of the method to a question hitherto, I believe, unattempted, I proceed to investigate the equation of the connector of the points in which the tangents at a node again meet a plane quartic curve. The only case in which this equation can be formed by ordinary methods is when the equation to the pair

of tangents in question breaks up into factors, viz., when the node is, say, $x=0$, $y=0$, $z=\Delta/\sin C$, and the line $x=0$ is one of the tangents. The equation is then of the form

$$a'x^4+b'y^4+6g'z^2x^2+6h'x^2y^2+12i'x^2yz+12j'y^2zx+12k'z^2xy$$
$$+4a_2'x^3y+4a_3'x^3z+4b_1'y^3x+4b_3'y^3z=0,$$

and the connector referred to is found, without difficulty, to be

$$4(2a'b_3'k'^3+3h'b_3'g'^2k'-4g'a_2'b_3'k'^2-g'^3b_1'b_3'-6b'g'^2i'k'+3b'j'g'^3+4b'a_3'g'k'^2)x'$$
$$+(8a_3'k'^3-12i'g'k'^2+6j'g'^2k'-b_3'g'^3)(b'y'+4b_3'z')=0.$$

To proceed to the general case:

If u is a quartic having a double point (xyz), the tangents at this point will be $lx'+my'+nz'$, subject to (8), Vol. IX., p. 228.

$$mn\,x\frac{d^2u}{dx^2}+nl\,y\frac{d^2u}{dy^2}+lm\,z\frac{d^2u}{dz^2}=0,$$

or
$$\lambda a+\mu b+\nu c=0,$$

where
$$\lambda=mn\,x\;\ldots\;4\,.\,3a=\frac{d^2u}{dx^2}\;\ldots\;;$$

viz., these tangents are two in number, as is well known, and meet the curve again in two points, the connector of which it is proposed to determine. Let $(x'y'z')$ be either of these points, then (6),

$$x'D^4u=xD^4u-4\,(m\gamma-n\beta)\,D^3u,$$

since for either tangent at the double point D^2u vanishes, as well as u and Du.

The term $(m\gamma-n\beta)\,D^3u$ may be advantageously calculated as $Dx\times D\,.\,D^2u$ by the formula (11); thus $(q=1,\ q'=2)$,

$$-2xyz\,(m\gamma-n\beta)\,D^3u=\Delta^2\lambda\,\frac{dD^2u}{dx}\text{ or }\tfrac{1}{2}\Delta^2\lambda D^2\frac{du}{dx}$$
$$-\Delta\left(\lambda ax\frac{dD^2u}{dx}+\mu\beta x\frac{dD^2u}{dy}+\nu\gamma x\frac{dD^2u}{dz}\right),$$

$$-x^2y^2z^2\,(m\gamma-n\beta)\,D^3u$$
$$\begin{aligned}=\quad&\Delta^2\lambda\,\{-\Delta^2(\lambda a_1+\mu b_1+\nu c_1)+2\Delta\,(\lambda aa+\mu\beta h+\nu\gamma g)\}\\ &+\Delta x\lambda a\,\{\;\Delta^2(\lambda a_1+\mu b_1+\nu c_1)-2\Delta\,(\lambda aa+\mu\beta h+\nu\gamma g)\}\\ &+\Delta x\mu\beta\,\{\;\Delta^2(\lambda a_2+\mu b_2+\nu c_2)-2\Delta\,(\lambda ah+\mu\beta b+\nu\gamma f)\}\\ &+\Delta x\nu\gamma\,\{\;\Delta^2(\lambda a_3+\mu b_3+\nu c_3)-2\Delta\,(\lambda ag+\mu\beta f+\nu\gamma c)\},\end{aligned}$$

$a \dots h$ being $\frac{d^2u}{dx^2} \dots\dots \frac{d^2u}{dx\,dy}$ each divided by 12, and $a_1 \dots c_3$ being $\frac{da}{dx} \dots\dots \frac{dc}{dz}$ each divided by 2, Δ standing for $\alpha x+\beta y+\gamma z$, $\alpha=\sin A \dots$.

Adding this development multiplied by 4 to that of $x^2y^2z^2D^4u$ multiplied by x, viz. (p. 230),

$$x\Delta^4\,(\lambda^2a'+\mu^2b'+\nu^2c'+2\mu\nu f'+2\nu\lambda g'+2\lambda\mu h')$$
$$-4x\Delta^3\,\{\lambda\alpha\,(\lambda a_1+\mu b_1+\nu c_1)+\mu\beta\,(\lambda a_2+\dots)+\nu\gamma\,(\lambda a_3+\dots)\}$$
$$+4x\Delta^2\,(\lambda^2\alpha^2a+\mu^2\beta^2b+\nu^2\gamma^2c+2\mu\nu\beta\gamma f+2\nu\lambda\gamma\alpha g+2\lambda\mu\alpha\beta h),$$

since $\lambda a+\mu b+\nu c$ vanishes for either of the tangents at the double point, and all the terms which follow in D^4u vanish as the conditions of (xyz) being a double point, there results

$$x^2y^2z^2x'D^4u = \Delta^4\,(\lambda^2xa'+\dots+2\lambda\mu xh'-4\lambda^2a_1-4\lambda\mu b_1-4\nu\lambda c_1)$$
$$+8\Delta^3\lambda\,(\lambda\alpha a+\mu\beta h+\nu\gamma g)$$
$$-4\Delta^2x\,(\lambda^2\alpha^2a+\mu^2\beta^2b+\nu^2\gamma^2c+2\mu\nu\beta\gamma f+2\nu\lambda\gamma\alpha g+2\lambda\mu\alpha\beta h).$$

This being multiplied by abc, and substituting for λ^2a, μ^2b, ν^2c, $-(\lambda\mu b+\nu\lambda c)$, $-(\lambda\mu a+\mu\nu c)$, $-(\nu\lambda a+\mu\nu b)$ respectively, in virtue of the fundamental relation, the right side of the equation above will contain only $\mu\nu$, $\nu\lambda$, $\lambda\mu$; so that, writing Δ^2k for $abcx^2y^2z^2D^4u$, it becomes

$$0 = kx'+a\,\{4bc\,(b\gamma^2+c\beta^2-2f\beta\gamma)\,x+\Delta^2\,(-b^2c'-c^2b'+2bcf')\,x\}\,\mu\nu$$
$$+b\,\{4ca\,(c\alpha^2+a\gamma^2-2g\gamma\alpha)\,x+\Delta^2\,(-c^2a'-a^2c'+2cag')\,x$$
$$+8\Delta ca\,(g\gamma-c\alpha)+4\Delta^2c\,(ca_1-ac_1)\}\,\nu\lambda$$
$$+c\,\{4ab\,(a\beta+b\alpha^2-2h\alpha\beta)\,x+\Delta^2\,(-a^2b'-b^2a'+2abh')\,x$$
$$+8\Delta ab\,(h\beta-b\alpha)+4\Delta^2b\,(ba_1-ab_1)\}\,\lambda\mu \quad\dots(33).$$

Similarly, interchanging letters and suffixes,

$$0=ky'+a\,\{4bc\,(b\gamma^2+c\beta^2-2f\beta\gamma)\,y+\Delta^2\,(-b^2c'-c^2b'+2bcf')\,y$$
$$+8\Delta bc\,(f\gamma-c\beta)+4\Delta^2c\,(cb_2-bc_2)\}\,\mu\nu$$
$$+b\,\{4ca\,(c\alpha^2+a\gamma^2-2g\gamma\alpha)\,y+\Delta^2\,(-c^2a'-a^2c'+2cag')\,y\}\,\nu\lambda$$
$$+c\,\{4ab\,(a\beta^2+b\alpha^2-2h\alpha\beta)\,y+\Delta^2\,(-a^2b'-b^2a'+2abh')\,y$$
$$+8\Delta ab\,(h\alpha-a\beta)+4\Delta^2a\,(ab_2-ba_2)\}\,\lambda\mu \quad\dots(34),$$

Considering now the other terms multiplied by $2\Delta^2$, their sum is readily found to be, when so multiplied,

$$2\Delta^2 abc\,(ayzL'+hzxM'+gxyN')\;\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii.)}.$$

The other terms containing L', M', N' are not divisible by abc until their values are substituted, when they are found to be equal to

$$\left.\begin{aligned}&abca'x\{[b^2(ca_3)y-c^2(ab_2)z]x+(gz-hy)[b(bc_3)y+c(bc_2)z]\}\\ &+a^2bcb'y^2\{(fz-hx)(ca_3)+a(bc_3)y-c(ca_2)z\}\\ &+a^2bcc'z^2\{(fy-gx)(ba_2)+b(ab_3)y-a(bc_2)z\}\\ &+2a^2bcf'yz\{b(ca_3)z-c(ab_2)y\}\\ &+2a^2bcg'z\{b(bc_3)yz+c[(bc_2)z^2+(ab_2)x^2]\}\\ &+2a^2bch'y\{c(cb_2)yz+b[(ac_3)x^2+(cb_3)y^2]\}\end{aligned}\right\}\ldots\text{(iv.)}.$$

The four parts (i.)—(iv.) of the coefficient of x' are now seen to be divisible by $\Delta^2 abc$; after which, and substitution of their values for L', M', N' in (iii.), the equation of the node-tangential line comes out in the invariant form,

$$\left.\begin{aligned}x'\,\Big|\,&8a^2bcf\\ &+4a\{b(fz-hx)(ca_3)+c(gx-fy)(ab_2)+ab(bc_3)y-ca(bc_2)z\\ &\qquad-x^2(ca_3)(ab_2)-y^2(ab_2)(bc_3)-z^2(bc_2)(ca_3)\}\\ &+a'x\{-2ax(b^2gy+c^2hz)\\ &\quad+[b^2(ca_3)y-c^2(ab_2)z]x+(gz-hy)[yb(bc_3)+zc(bc_2)]\}\\ &+ab'y\{-2y(c^2z+a^2x)+y(fz-hx)(ca_3)+y^2a(bc_3)-yza(bc_2)\}\\ &+ac'z\{-2z(a^2x+b^2y)+z(fy-gx)(ba_2)+yza(bc_3)-z^2a(bc_2)\}\\ &+2af'yz\{2abc+b(ca_3)z-c(ab_2)y\}\\ &+2ag'z\{2achx+b(bc_3)yz+c(bc_2)z^2+c(ab_2)x^2\}\\ &+2ah'y\{2abgx-b(ca_3)x^2-b(bc_3)y^2-c(bc_2)yz\}\\ +y'\,\Big|\,&8ab^2cg\\ &+\ldots\ldots\\ +z'\,\Big|\,&8abc^2h\\ &+\ldots\ldots\qquad\qquad\qquad\qquad\qquad\qquad=0\end{aligned}\right\}\ldots(36),$$

an equation of the fifth degree in the coefficients of u, of the tenth in xyz.

In using this general form to obtain a particular result, *e.g.*, when the double point is $x=0$, $y=0$, the highest powers of x, y must be successively made to vanish until the result is reduced to one in which the residue can be cleared of x, y by division. Thus, to obtain the

particular case investigated prior to entering on the general question, it is composed of the terms of the general form in which $x^3y^3z^4$ is a common divisor. Thus the first term in the coefficient of x', viz., $8a'b'_3k'^3$, comes from $xya'hc^2b_2$, and so on.

If the point (xyz) should be a cusp, the two tangents coincide in a single one, the equation to which may be written

$$ax'+hy'+gz'=0,$$

or

$$hx'+by'+fz'=0,$$

or

$$gx'+fy'+cz'=0,$$

indifferently. Using the first form in the equation (33) above, viz., substituting a, h, g for l, m, n, and similarly substituting h, b, f for l, m, n in (34); g, f, c for l, m, n in (35), and recollecting that $\mu\nu = lmn\,.\,lyz$, $\nu\lambda = lmn\,.\,mzx$, $\lambda\mu = lmn\,.\,nxy$; also that at a cusp $gh = af$, $hf = bg$, $fg = ch$, it is easily found that the coordinates of the point $(x'y'z')$, in which the cuspidal tangent again meets the quartic u, are determined by

$$\begin{aligned} x':y':z' = x\,\{&4a^2bcf-abc\,(fa_1x+gb_1y+hc_1z)+af\,(bca'x^2+cab'y^2+abc'z^2) \\ &+2abc\,(af'yz+hg'zx+gh'xy)\} \\ :y\,\{&4ab^2cg-abc\,(fa_2x+gb_2y+hc_2z)+bg\,(bca'x^2+cab'y^2+abc'z^2) \\ &+2abc\,(hf'yz+bg'zx+fh'xy)\} \\ :z\,\{&4abc^2h-abc\,(fa_3x+gb_3y+hc_3z)+ch\,(bca'x^2+cab'y^2+abc'z^2) \\ &+2abc\,(gf'yz+fg'zx+ch'xy)\}. \end{aligned}$$

Of course the equation (36) becomes in this case the tangent at the tangential point of the cusp.

April 2nd, 1885.

J. W. L. GLAISHER, Esq., F.R.S., President, in the Chair.

Dr. R. Stawell Ball, F.R.S., Astronomer-Royal for Ireland, and Baboo Syama Charan Basu, B.A., of Bhowanipore, were elected members.

The following communications were made :—

New Relations between Bipartite Functions and Determinants, with a Proof of Cayley's Theorem in Matrices: Dr. T. Muir.

On Eliminants, and Associated Roots: E. B. Elliott, M.A.

On Five Properties of Certain Solutions of a Differential Equation of the Second Order: Dr. Routh, F.R.S.

On the Arguments of Points on a Surface: R. A. Roberts, M.A.

On Congruences of the Third Order and Class: Dr. Hirst, F.R.S.

The following presents were received:—

"The Common Sense of the Exact Sciences," by the late W. K. Clifford, 8vo; London, 1885.

"Curve Tracing in Cartesian Coordinates," by Prof. W. Woolsey Johnson, 8vo; New York, 1884.

"Johns Hopkins University Circulars," Vol. iv., Nos. 37 and 38, March, 1885.

"Memoirs of the National Academy of Sciences," Vol. ii., 1883, 4to; Washington, 1884.

"Archives Néerlandaises des Sciences Exactes et Naturelles," T. xix., L. 4 and 5.

"Bulletin de la Société Mathématique de France," Tome xiii., No. 2.

"Tidsskrift for Mathematik," V. Raekke, 1 Aargang, Nos. 1—6.

"Bulletin des Sciences Mathématiques et Astronomiques," T. ix., March, 1885.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. ix., St. 3.

"Teorema per la Quadratura del Circolo," di Ugo Amerighi, 8vo; Firenze, 1885.

"Acta Mathematica," v., 2, 3, and 4.

"Journal für Mathematik," Bd. xcviii., Heft 2; March, 1885.

"Atti della R. Accademia dei Lincei,—Rendiconti," Vol. i., Fasc. 7; Roma, March, 1885.

"Annales de l'Ecole Polytechnique de Delft," 2 liv.; Leide, 1885.

On Eliminants, and Associated Roots. By E. B. Elliott, M.A.

[*Read April 2nd,* 1885.]

1. The object of the following paper is to obtain some more general results akin to and including the well-known simple and elegant theorems due to Professor Sylvester and others, which connect the common root of two quantics with the differential coefficients of their resultant, the repeated root of a single quantic with those of the discriminant whose vanishing expresses its existence, &c. The method adopted is that of Salmon's *Higher Algebra*, § 92.

Let $$u_x \equiv a_1\psi_1(x)+a_2\psi_2(x)+\ldots+a_m\psi_m(x),$$

and $$v_y \equiv b_1\chi_1(y)+b_2\chi_2(y)+\ldots+b_n\chi_n(y),$$

and let the condition in the coefficients, which expresses that a value of x satisfying $u = 0$, and one of y which makes $v = 0$, are connected by a given relation
$$\phi(x, y) = 0,$$
be
$$F(a_1, a_2, \dots a_m, b_1, b_2, \dots b_n) = 0.$$

Also let δx, δy, $\delta a_1, \dots \delta b_n$ be infinitesimal increments given to the variables and coefficients, restricted only by the conditions that the equations $u = 0$, $v = 0$, $\phi = 0$ still hold. It is a consequence that the relation $F = 0$ still holds equally. In other words, retaining only first powers of the increments, the three limitations of arbitrariness,

$$\psi_1(x)\,\delta a_1 + \dots + \psi_m(x)\,\delta a_m + \frac{du_x}{dx}\,\delta x = 0,$$

$$\chi_1(y)\,\delta b_1 + \dots + \chi_n(y)\,\delta b_n + \frac{dv_y}{dy}\,\delta y = 0,$$

$$\frac{d\phi}{dx}\,\delta x + \frac{d\phi}{dy}\,\delta y = 0,$$

must produce as a consequence

$$\frac{dF}{da_1}\,\delta a_1 + \dots + \frac{dF}{da_m}\,\delta a_m + \frac{dF}{db_1}\,\delta b_1 + \dots + \frac{dF}{db_n}\,\delta b_n = 0.$$

Consequently, for certain values of three undetermined multipliers λ, μ, ν, we must have simultaneously

$$\frac{\dfrac{dF}{da_1}}{\psi_1(x)} = \frac{\dfrac{dF}{da_2}}{\psi_2(x)} = \dots = \frac{\dfrac{dF}{da_m}}{\psi_m(x)} = \lambda,$$

$$\frac{\dfrac{dF}{db_1}}{\chi_1(y)} = \frac{\dfrac{dF}{db_2}}{\chi_2(y)} = \dots = \frac{\dfrac{dF}{db_n}}{\chi_n(y)} = \mu,$$

$$\nu\,\frac{d\phi}{dx} - \lambda\,\frac{du_x}{dx} = 0,$$

$$\nu\,\frac{d\phi}{dy} - \mu\,\frac{dv_y}{dy} = 0,$$

altogether $m+n+2$ relations, of which the last two tell us that the value of the ratio $\lambda : \mu$ of the common values of the first and second sets of equal fractions is

$$\frac{d\phi}{dx} \cdot \frac{dv_y}{dy} : \frac{d\phi}{dy} \cdot \frac{du_x}{dx}.$$

2. It becomes desirable to interpret special cases of these general results. Firstly, then, let us take the case of direct elimination of a single variable x between two equations

$$u_x \equiv a_1\psi_1(x)+a_2\psi_2(x)+\ldots+a_m\psi_m(x)=0,$$

and $$v_x \equiv b_1\chi_1(x)+b_2\chi_2(x)+\ldots+b_n\chi_n(x)=0;$$

that is to say, let us take

$$\phi(x, y) \equiv x-y,$$

and let us now write the result of eliminating

$$E(a_1, a_2, \ldots a_m, b_1, b_2, \ldots b_n)=0.$$

Our conclusion is that the common x which satisfies $u_x=0$ and $v_x=0$ satisfies also the $m+n-1$ equations

$$\frac{dv_x}{dx}\cdot\frac{\psi_1(x)}{\frac{dE}{da_1}}=\frac{dv_x}{dx}\cdot\frac{\psi_2(x)}{\frac{dE}{da_2}}=\ldots=\frac{dv_x}{dx}\cdot\frac{\psi_m(x)}{\frac{dE}{da_m}}$$

$$=-\frac{du_x}{dx}\cdot\frac{\chi_1(x)}{\frac{dE}{db_1}}=-\frac{du_x}{dx}\cdot\frac{\chi_2(x)}{\frac{dE}{db_2}}=\ldots=-\frac{du_x}{dx}\cdot\frac{\chi_n(x)}{\frac{dE}{db_n}}.$$

To particularise still further, let $\psi_1(x), \ldots \psi_m(x), \chi_1(x), \ldots \chi_n(x)$ be powers of x. The two lines of equalities then give $m-1$ and $n-1$, immediate expressions for the common value of x, as ratios of differential coefficients of the eliminant or as roots of such ratios. When the powers of x are two sets of successive powers, so that u_x and v_x are complete binary quantics expressed in their natural form, the values thus given are those of Salmon's *Higher Algebra*, § 92.

3. Another course that may be taken by way of deducing more special results from the general ones of § 1 will be, keeping $\phi(x, y)$ general, to restrict the forms of the $m+n$ functions $\psi(x)$ and $\chi(y)$. For instance, we obtain immediately that, if

$$K=0$$

is the condition in the coefficients that x, a root of

$$U_x \equiv a_1x^{r_1}+a_2x^{r_2}+\ldots+a_mx^{r_m}=0,$$

and y, one of $$V_y \equiv b_1y^{s_1}+b_2y^{s_2}+\ldots+b_ny^{s_n}=0,$$

are connected by a relation

$$\phi(x, y)=0,$$

then equivalent values of these roots are given by pairs of the $m+n$ equations

$$\frac{dK}{da_1} = \lambda x^{r_1}, \quad \frac{dK}{da_2} = \lambda x^{r_2}, \quad \ldots\ldots, \quad \frac{dK}{da_m} = \lambda x^{r_m},$$

$$\frac{dK}{db_1} = \mu y^{s_1}, \quad \frac{dK}{db_2} = \mu y^{s_2}, \quad \ldots\ldots, \quad \frac{dK}{db_n} = \mu y^{s_n},$$

where

$$\lambda \frac{d\phi}{dy} \cdot \frac{dU_x}{dx} = \mu \frac{d\phi}{dx} \cdot \frac{dV_y}{dy}.$$

For the case expressed by

$$r_1 = r_2 + 1 = r_3 + 2 = \ldots = r_m + m - 1,$$

$$s_1 = s_2 + 1 = s_3 + 2 = \ldots = s_n + n - 1,$$

this, again, is another immediate generalisation of Salmon, § 92.

4. Returning to the general theorems of § 1, a case of special interest and exceptional nature is that in which the two sets of coefficients $a_1, a_2, \ldots, b_1, b_2, \ldots$ are identical. In such a case the numbers of functions $\psi(x)$ and $\chi(y)$ will generally be equal; but other cases will be included.

Suppose, in § 1, that n is equal to m. We are told that, for all positive integral values of r not exceeding m,

$$\frac{dF}{da_r} = \lambda \psi_r(x), \quad \text{and} \quad \frac{dF}{db_r} = \mu \chi_r(y),$$

where

$$\lambda \frac{d\phi}{dy} \cdot \frac{du_x}{dx} = \mu \frac{d\phi}{dx} \cdot \frac{dv_y}{dy}.$$

Now suppose that, on replacing $b_1, b_2, \ldots b_m$ by $a_1, a_2, \ldots a_m$ respectively, the function

$$F(a_1, a_2, \ldots a_m, b_1, b_2, \ldots b_m),$$

becomes

$$f(a_1, a_2, \ldots a_m),$$

then, for each value of r,

$$\frac{dF}{da_r} + \frac{dF}{db_r} \text{ becomes } \frac{df}{da_r}.$$

Consequently, we obtain that, if

$$f(a_1, a_2, \ldots a_m) = 0$$

be the condition in the coefficients that

$$u_x \equiv a_1 \psi_1(x) + a_2 \psi_2(x) + \ldots + a_m \psi_m(x) = 0,$$

and $$u'_y \equiv a_1\chi_1(y) + a_2\chi_2(y) + \ldots + a_m\chi_m(y) = 0$$

are satisfied by an x and a y connected by

$$\phi(x, y) = 0,$$

it follows that $$\frac{df}{da_1} = \lambda\psi_1(x) + \mu\chi_1(y),$$

$$\frac{df}{da_2} = \lambda\psi_2(x) + \mu\chi_2(y),$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

and $$\frac{df}{da_m} = \lambda\psi_m(x) + \mu\chi_m(y);$$

where λ and μ are in the ratio

$$\frac{d\phi}{dx} \cdot \frac{du'_y}{dy} : \frac{d\phi}{dy} \cdot \frac{du_x}{dx}.$$

No modification is necessary if m and n be unequal. If, for instance, $m > n$ we have only to consider in these results that $\chi_{n+1}(y), \ldots \chi_m(y)$ are identically zero.

5. Let now, in this last, u_x and u'_y be the same functions of x and y respectively. We conclude at once that, if x and y be two roots of the equation

$$u_x \equiv a_1\psi_1(x) + a_2\psi_2(x) + \ldots + a_m\psi_m(x) = 0,$$

which are connected by $\phi(x, y) = 0,$

and, if $$F(a_1, a_2, \ldots a_m) = 0$$

be the condition in the coefficients expressive of the fact, then

$$\frac{dF}{da_1} = \lambda\psi_1(x) + \mu\psi_1(y),$$

$$\frac{dF}{da_2} = \lambda\psi_2(x) + \mu\psi_2(y),$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\frac{dF}{da_m} = \lambda\psi_m(x) + \mu\psi_m(y),$$

where
$$\frac{\lambda}{\mu} = \frac{\dfrac{d\phi}{dx} \cdot \dfrac{du_y}{dy}}{\dfrac{d\phi}{dy} \cdot \dfrac{du_x}{dx}},$$

which ratio may be written $\quad -\dfrac{du_y}{du_x}.$

In particular, give u_x the form

$$U_x \equiv a_1 x^{r_1} + a_2 x^{r_2} + \dots + a_m x^{r_m};$$

and write L for the form then assumed by $\mathbf{F}\,(a_1, a_2, \dots a_m)$.

We deduce m equations, of which the first

$$\frac{dL}{da_1} = \lambda x^{r_1} + \mu y^{r_1}$$

is representative, and conclude that, if only $r_1, r_2, \dots r_n$ are commensurable quantities, $\dfrac{dL}{da_1}, \dfrac{dL}{da_2}, \dots \dfrac{dL}{da_m}$ are terms of a recurring series. In particular, if $r_1, r_2, \dots r_m$ be positive integers, the scale of relation of this recurring series is $1-(x+y)+xy$; and the various differential coefficients of L, arranged in order as above, are the $r_1^{\text{th}}, r_2^{\text{th}}, \dots r_m^{\text{th}}$ terms of the series. More particularly still, if U_x be the ordinary complete binary $(m-1)$-ic, $\quad a_1 x^{m-1} + a_2 x^{m-2} + \dots + a_m,$

those differential coefficients are the first m terms of a recurring series whose scale of relation is

$$xy - (x+y) + 1 = 0.$$

Since the ratio $\lambda : \mu$ is given, one known term will determine our recurring series completely in terms of x and y.

6. Desiring to proceed to the case of two equal roots of the same equation $u_x = 0$, two courses are open to us. We may take the results of the last article, specialise them by assuming

$$\phi\,(x, y) \equiv x - y + \delta,$$

and proceed to the limit by making δ infinitesimal. [Merely to replace $\phi\,(x, y)$ by $x-y$ would, for several apparent reasons, be nugatory.] Or, remembering that equal roots of $u_x = 0$ are roots also of $\dfrac{du_x}{dx} = 0$, we may apply § 4, taking $u'_y \equiv \dfrac{du_y}{dy}$, and $\phi\,(x, y) \equiv x - y$. The results obtained by the two processes are, as they should be,

identical. The latter is the one here chosen for exhibition. Briefly stated, it tells us that, $\Delta(a_1, a_2, \dots a_m) = 0$

being the condition that an x simultaneously satisfies

$$u_x \equiv a_1\psi_1(x) + a_2\psi_2(x) + \dots + a_m\psi_m(x) = 0,$$

and
$$\frac{du_x}{dx} \equiv a_1\psi_1'(x) + a_2\psi_2'(x) + \dots + a_m\psi_m'(x) = 0,$$

then
$$\frac{d\Delta}{da_1} = \lambda\psi_1(x) + \mu\psi_1'(x),$$

$$\frac{d\Delta}{da_2} = \lambda\psi_2(x) + \mu\psi_2'(x),$$

$$\dots \quad \dots \quad \dots \quad \dots$$

$$\frac{d\Delta}{da_m} = \lambda\psi_m(x) + \mu\psi_m'(x),$$

simultaneously, where λ and μ satisfy

$$\lambda\frac{du_x}{dx} + \mu\frac{d^2u_x}{dx^2} = 0;$$

which last expresses, since for the particular value x in question $\frac{du_x}{dx} = 0$, that, in general, $\frac{\mu}{\lambda} = 0$. We can neglect, therefore, μ in comparison with λ, and so arrive at the equalities

$$\frac{\frac{d\Delta}{da_1}}{\psi_1(x)} = \frac{\frac{d\Delta}{da_2}}{\psi_2(x)} = \dots = \frac{\frac{d\Delta}{da_m}}{\psi_m(x)} = \lambda.$$

Of this general theorem with regard to equal roots, the case in which the functions $\psi(x)$ are successive powers of x is, of course, a familiar one. (Salmon's *Higher Algebra*, § 109.)

7. Similar theories to all the above, though more cumbrous in form, can be developed with regard to functions of higher numbers of variables. Let it suffice here to state the results corresponding to the comprehensive ones of the first article for the case when u and v involve each two variables.

Suppose $u \equiv a_1\psi_1(x, x') + a_2\psi_2(x, x') + \dots + a_m\psi_m(x, x'),$

and $v \equiv b_1\chi_1(y, y') + b_2\chi_2(y, y') + \dots + b_n\chi_n(y, y'),$

and let the condition in the coefficients, which expresses that certain values of x, x', y, y', which make $u = 0$ and $v = 0$, are connected by three given relations

$$\phi_1(x, x', y, y') = 0,$$
$$\phi_2(x, x', y, y') = 0,$$
$$\phi_3(x, x', y, y') = 0,$$

be
$$F(a_1, a_2, \dots a_m, b_1, b_2, \dots b_n) = 0;$$

then shall
$$\frac{\frac{dF}{da_1}}{\psi_1(x, x')} = \frac{\frac{dF}{da_2}}{\psi_2(x, x')} = \dots = \frac{\frac{dF}{da_m}}{\psi_m(x, x')} = \lambda,$$

and
$$\frac{\frac{dF}{db_1}}{\chi_1(y, y')} = \frac{\frac{dF}{db_2}}{\chi_2(y, y')} = \dots = \frac{\frac{dF}{db_n}}{\chi_n(y, y')} = \mu,$$

where λ and μ are connected by the result of eliminating ν_1, ν_2, ν_3 from the four equations

$$\lambda \frac{du}{dx} = \nu_1 \frac{d\phi_1}{dx} + \nu_2 \frac{d\phi_2}{dx} + \nu_3 \frac{d\phi_3}{dx},$$
$$\lambda \frac{du}{dx'} = \nu_1 \frac{d\phi_1}{dx'} + \nu_2 \frac{d\phi_2}{dx'} + \nu_3 \frac{d\phi_3}{dx'},$$
$$\mu \frac{dv}{dy} = \nu_1 \frac{d\phi_1}{dy} + \nu_2 \frac{d\phi_2}{dy} + \nu_3 \frac{d\phi_3}{dy},$$
$$\mu \frac{dv}{dy'} = \nu_1 \frac{d\phi_1}{dy'} + \nu_2 \frac{d\phi_2}{dy'} + \nu_3 \frac{d\phi_3}{dy'},$$

that is to say, by the relation

$$\lambda \left\{ \frac{du}{dx} \cdot \frac{d(\phi_1, \phi_2, \phi_3)}{d(x', y, y')} - \frac{du}{dx'} \cdot \frac{d(\phi_1, \phi_2, \phi_3)}{d(x, y, y')} \right\}$$
$$+ \mu \left\{ \frac{dv}{dy} \cdot \frac{d(\phi_1, \phi_2, \phi_3)}{d(x, x', y')} - \frac{dv}{dy'} \cdot \frac{d(\phi_1, \phi_2, \phi_3)}{d(x, x', y)} \right\} = 0.$$

On Congruences of the Third Order and Class.

By Dr. T. ARCHER HIRST, F.R.S.

[*Read April 2nd*, 1885.]

1. In the *Monats Bericht* of the Academy of Berlin, for the 17th of January, 1878, Kummer had occasion to study the properties of two congruences of the third order and class, having one and the same focal surface of the eighth order and class. A year ago, when engaged on Cremonian Congruences,* I was led to consider two associated ones of the third order and class, which, at first sight, appeared to be identical with those of Kummer. Further examination, however, soon proved them to be varieties of the latter which have not hitherto, I believe, been noticed. I propose to examine them in the present paper, and, in particular, to show in what manner they presented themselves to me as degenerate cases of Cremonian Congruences of the fifth order and fourth class.

2. Such a congruence (5, 3) is generated by joining the corresponding points of two, arbitrarily situated, planes α and β, between which a Cremonian Correspondence of the third order has been established.† The planes α and β are singular ones, containing congruence-curves a_4 and b_4, respectively, of the sixth order and fourth class.‡ The intersection $\overline{\alpha\beta}$ is a triple congruence-ray and, at the same time, a triple tangent of each of the above congruence-curves; the points of contact, in each plane, being the intersections of $\overline{\alpha\beta}$ with the fourth-class cubic, in the other plane, which corresponds to that line.§

The double points A_2 and B_2 of the above cubics $\bar{a}^3$ and $\bar{b}^3$ are at the principal double points of the cubic correspondence. They are the vertices of quadric congruence-cones A_2^2 and B_2^2, passing through all the five principal points of the correspondence, which are situated in β and α, respectively.‖ The four pairs of associated principal single points of this correspondence—say, A_1', B_1'; A_1'', B_1''; A_1''', B_1''', and A_1'''', B_1''''—are centres of four pairs of congruence-pencils; the planes of each pair intersect in the line passing through both centres, and pass through the principal double points. The plane of each of the eight pencils, in short, passes through the principal line which corresponds to its centre.

3. Besides α and β and the eight, above described, there are two

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* *Proceedings of the London Mathematical Society*, Vol. XIV., pp. 259—301. Hereafter the articles of this paper will be referred to in the abridged form, C. C.

† C. C., Arts. 3 and 6. ‡ C. C., Art. 7. § C. C., Art. 8. ‖ C. C., Art. 9.

other singular planes, each of which passes through *both* the principal double points A_2 and B_2, and intersects α and β in a pair of lines which correspond to one another, point by point. Each of these planes, therefore, contains a congruence-conic which touches α and β at points on $\bar{a}^3$ and $\bar{b}^3$.

4. The several tangents of these two congruence-conics, and the rays of each of the above four pairs of congruence-pencils, belong, in reality, to a system of quadric reguli,* whose generators, in the aggregate, constitute the congruence (5, 3) under consideration. The two directrices of each regulus of the system being corresponding rays of the pencils $A_2(\alpha)$ and $B_2(\beta)$, one generator thereof lies in α, and another in β; these generators join, obviously, the intersections, with $\overline{\alpha\beta}$, of the two directrices of the regulus, with their corresponding points on $\bar{a}^3$ and $\bar{b}^3$. The quadric surfaces, therefore, on which the several reguli of the system are situated, all pass through the principal points A_2 and B_2, and touch the planes α and β at points which lie on the curves $\bar{a}^3$ and $\bar{b}^3$.

5. The second, or conjugate system of reguli, situated on the above quadrics, form, in the aggregate, another congruence (5, 3), which has the same singular points and planes—in short, the same focal surface as the one already described. But *this* congruence is not Cremonian, that is to say, its rays do not determine a bi-rational correspondence between any two of its singular planes; A_2 and B_2 are here simply centres of congruence-pencils situated in α and β.†

6. The common focal surface of the above two congruences is the envelope of the system of quadric-surfaces already referred to. It is of the twelfth order and eighth class; touches the planes α and β along the cubics $\bar{a}^3$ and $\bar{b}^3$, and, besides this, cuts them in the fourth class, sextic congruence-curves a_4 and b_4.‡

7. A_2 and B_2 are, in general, quadruple points of the focal surface; the tangent cone at each being the envelope of the plane which connects a ray of $A_2(\alpha)$, or $B_2(\beta)$, with the point in which its corresponding ray of $B_2(\beta)$, or $A_2(\alpha)$, intersects, for the second time, the principal conic in β, or α.§ This cone is well known to be of the third class and fourth order; it touches the plane α, or β, twice, and has triple contact with the quadric congruence-cone A_2^2, or B_2^2.

It is important for what follows to notice that, by the formulæ of Plucker, each of these cones possesses *three cuspidal edges*.

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8. Without further discussing the properties of the above focal

* Conf. C. C., Art. 32. † Conf. C. C., Art. 36.
‡ C. C., Arts. 10, 13, and 15. § Conf. C. C., Art. 36 (c).

surface, I pass, at once, to the degenerate case of the congruence (5, 3) which was referred to in Art. 1.

It presents itself when the planes α and β are so placed that each of two points C and D, on $\overline{\alpha\beta}$, becomes self-correspondent. Two congruences (1, 0), consisting of rays issuing from these points, may then be detached from the congruence (5, 3). This done, the residual congruence (3, 3), say **C**, will still have the two quadric congruence-cones A_2^2 and B_2^2, of Art. 2, and the four pairs of congruence-pencils with centres A_1', B_1'; A_1'', B_1''; A_1''', B_1'''; and A_1'''', B_1''''. The planes α and β, however, will now contain congruence-conics a_2 and b_2, having $\overline{\alpha\beta}$ for a common tangent, and, instead of the two congruence-conics, described in Art. 3, we shall now have two congruence-pencils, with centres C' and D', supplemented by two other pencils, with centres C and D, respectively, and situated in planes γ and δ, each of which touches *both* the conics a_2 and b_2.* Included in the system of quadric reguli whose generators, in the aggregate, constitute the congruence **C** under consideration, we have now, in short, two new pencil-pairs,—one with the centres C and C' and the planes γ and (A_2CB_2), the other with the centres D and D' and the planes δ and (A_2DB_2), respectively.

9. The reguli, conjugate to those of the system just referred to, constitute, in the aggregate, a second congruence (3, 3), say $\mathbf{C}_1$, having the same singular points and planes as **C**.

In passing from **C** to $\mathbf{C}_1$, however, the planes of the pencil-pairs whose centres are A_1', B_1'; A_1'', B_1''; A_1''', B_1'''; A_1'''', B_1''''; C, C'; D, D' are interchanged, and α and β are now merely planes of congruence-pencils having their centres at A_2 and B_2 respectively.† The congruence $\mathbf{C}_1$, moreover, is no longer Cremonian; its rays establish, in fact, a (2, 2) correspondence between any two of its singular planes.

10. Each of the congruences **C** and $\mathbf{C}_1$ is self-reciprocal, as is also their common focal surface. The latter, which is, of course, the envelope of the system of quadric surfaces upon which the above-mentioned conjugate reguli are situated, is of the eighth order and class.

Besides touching the planes α and β along the fourth-class cubics $\overline{a}^3$ and $\overline{b}^3$, the focal surface likewise cuts them in the conics a_2 and b_2; and, correlatively, A_2 and B_2 are not only quadruple points of the focal surface, the cones of contact at which are of the third class (Art. 7),

* C. C., Art. 20.

† Conf. C. C., Art. 36, (c) and (d).

but they are likewise vertices of quadric cones, A_2^2 and B_2^2, circumscribed to the surface elsewhere.

The points of each of the six point-and-plane pairs, included in the system of enveloping quadrics, are nodes of the focal surface, the quadric tangent-cones at which touch both the planes of the pair; and correlatively these planes touch the focal surface along conics which pass through both the points of the pair. These conics form, obviously, the intersection of the point-and-plane pair with the next succeeding quadric of the system.

11. Generally, every two consecutive quadrics of the system referred to in the last Art. intersect, in a quartic curve—the characteristic—which passes through A_2 and B_2 and touches α and β. The tangent at each of these two points is a generator of one of the third class, quartic cones referred to in Art. 7, and in each of these two planes the characteristic touches one of the fourth-class cubics $\bar{a}^3$ and $\bar{b}^3$.

12. Two consecutive characteristics, in other words, three consecutive quadrics of the system in Art. 10, intersect, not only in A_2 and B_2, but in six other points situated on the *curve of regression* of the focal surface. At each of these six points the curve in question has three-pointic contact with one of the quadrics S^2 of the system. The three points, in fact, are the intersection of S^2 with (1) the two quadrics immediately preceding it, (2) the quadrics immediately preceding and following it, and (3) the two quadrics immediately succeeding it. The above six intersections of two consecutive characteristics, therefore, count as eighteen intersections of S^2 and the curve of regression.

But the points A_2 and B_2, as will be presently shown, are triple points on the curve in question; so that the latter intersects S^2, on the whole, in twenty-four points, and is consequently, like Kummer's curve of regression, of the *twelfth order*.

13. That A_2 and B_2 are, in reality, triple points on our curve of regression follows most readily from the fact, alluded to at the end of Art. 7, that the tangent-cone to the focal surface at each of these, its quadruple points, has three cuspidal edges. For the generators of these cones are clearly tangents, at A_2 and B_2, respectively, of the several characteristics passing through these points, and the existence of a cuspidal edge thereof obviously implies that of a characteristic of a similar stationary character,—that is to say, of one which intersects its consecutive characteristic in six points whereof one has come to coincide with A_2, or B_2.

14. The curve of regression of the focal surface, besides having triple points at A_2 and B_2, has, as we have seen in Art. 12, sextuple,

three-pointic contact with every quadric of the system in Art. 10. Six of the latter, however, break up into point-and-plane pairs. The planes of each of four of these degenerate quadrics pass through two associated principal single points of the cubic correspondence between α and β, as do also the two conics along which these planes touch the focal surface. One of these conics, moreover, passes through A_2 and the other through B_2; and each of them likewise passes through the three points in each of which its plane osculates the curve of regression.

The planes of each of the remaining two point-and-plane pairs of the system touch the focal surface along a pair of conics intersecting in C and C', in the one case, and in D and D', in the other. In both cases, one of the conics of the pair passes through the two principal points, A_2 and B_2, while the other, situated in γ or δ, passes through the four points in each of which its plane osculates the curve of regression.

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15. The preceding congruence (3, 3) differs from that described by Kummer in having the two singular points A_2 and B_2, and the two singular planes α and β. There is, however, a still more special congruence (3, 3) which merits a passing notice.

When the correspondence between α and β is such that the distance between two of the four principal single points in the former is equal to that between their associates in the latter, both pairs may be brought to coincide on $\overline{\alpha\beta}$; A_1''' and B_1''', for instance, in C, and A_1'''' and B_1'''' in D. When the generating planes are thus placed, however, each of the cubics $\overline{a}^3$ and $\overline{b}^3$, corresponding to $\overline{\alpha\beta}$, breaks up into three right lines, viz.: $\overline{CA_2}$, $\overline{DA_2}$, and $\overline{A_1'A_1''}$ in α, and $\overline{CB_2}$, $\overline{DB_2}$, $\overline{B_1'B_1''}$ in β. To the several points of $\overline{CA_2}$ and $\overline{CB_2}$ corresponds the point C solely, and to those of $\overline{DA_2}$ and $\overline{DB_2}$ the point D; while between $\overline{\alpha\beta}$ and $\overline{A_1'A_1''}$, as well as between $\overline{\alpha\beta}$ and $\overline{B_1'B_1''}$, a point-to-point correspondence exists, whereby the two congruence-conics a_2 and b_2 are generated. Each of these conics touches $\overline{\alpha\beta}$, as well as the three right lines into which the cubic of its plane has broken up.

16. The planes γ and δ of the congruence-pencils whose centres are C and D respectively, since they touch both the conics a_2 and b_2, (Art. 8), now coincide, respectively, with the planes (A_2CB_2) and (A_2DB_2); and at the same time the centres, C' and D', of the congruence-pencils which these planes formerly contained, now coincide, respectively, with C and D. In short, in place of two of the congruence pencil-pairs described in Art. 8, we have now two *doubled* pencils $\widehat{A_2CB_2}$ and [illegible]B_2.

These, as degenerate quadric reguli belonging to the system described in Arts. 4 and 8, coincide, moreover, with their conjugate reguli. In other words, the above doubled pencils, like the tangents of the two conics described in Art. 3, are common to the two associated congruences referred to in Arts. 5 and 9.

17. The system of quadric surfaces on which the above-mentioned reguli are situated, and whose envelope is the focal surface common to the congruences **C** and $\mathbf{C}_1$, now includes the two doubled planes (A_2CB_2) and (A_2DB_2), each bounded by a pair of right lines,—$\overline{A_2C}$, $\overline{B_2C}$ and $\overline{A_2D}$, $\overline{B_2D}$ respectively — which lie wholly on the focal surface.

On this surface the lines $\overline{A_1'A_1''}$ and $\overline{B_1'B_1''}$ also lie; along them, in fact, the surface touches the planes α and β, respectively, since each quadric of the enveloping system touches these planes at points situated on these lines (Art. 4).

18. The focal surface being, like the congruences **C** and $\mathbf{C}_1$, self-reciprocal, we might at once infer from the above that the tangent-planes to it at A_2, as well as those at B_2, are co-axal, the axes being generators, respectively, of the congruence-cones A_2^2 and B_2^2.

These results, however, are easily verified directly. The tangent-planes in question each connect a ray of $A_2(\alpha)$ or $B_2(\beta)$ with a generator of the congruence-cone A_2^2 or B_2^2 (Art. 7), and these rays and generators are now not only in (1, 1) correspondence with each other, but in each of the lines $\overline{A_2C}$ and $\overline{A_2D}$, or $\overline{B_2C}$ and $\overline{B_2D}$, two corresponding elements coincide. By a well-known theorem, therefore the planes of all other pairs of corresponding elements are co-axal. The axis, of course, is the intersection of the plane $(A_2A_1'B_1')$ with $(A_2A_1''B_1'')$, or of $(B_2A_1'B_1')$ with $(B_2A_1''B_1'')$, and from elementary considerations it can be readily shown that this intersection lies on the congruence-cone $A_2^2 \equiv A_2(B_2B_1'B_1''CD)$, or $B_2^2 \equiv B_2(A_2A_1'A_1''CD)$.

Finally, it may be observed that this axis, and the lines $\overline{A_2C}$, $\overline{A_2D}$, or $\overline{B_2C}$, $\overline{B_2D}$, represent, in the present case, the three stationary edges of the tangent-cone at A_2, or B_2, to the focal surface—which edges were found in Art. 13 to be likewise tangents, at the triple points, of the curve of regression on this focal surface.

On the Arguments of Points on a Surface.

By R. A. Roberts, M.A.

[*Read April 2nd,* 1885.]

It is known that, if a variable general curve of assigned degree meet a fixed curve in a plane, the points of intersection are connected by certain relations. These relations have been expressed by Clebsch in a well-known form involving certain integrals which depend upon the position of the points on the fixed curve. There are also similar relations connecting the points on a fixed twisted curve where it is intersected by a variable surface. Now, if we seek the extension of these results to the points of intersection of a fixed surface with variable curves, it is evident that we must have relations between several double integrals, as the position of a point on a surface depends on two variables. From the analogy of plane curves, we should be led to expect

$$\frac{dx\,dy}{\dfrac{dF}{dz}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

as the element of the fundamental double integral. This is a symmetrical expression, being equal to

$$\frac{dy\,dz}{\dfrac{dF}{dx}} = \frac{dz\,dx}{\dfrac{dF}{dy}} = \frac{dS}{\sqrt{\left\{\left(\dfrac{dF}{dx}\right)^2 + \left(\dfrac{dF}{dy}\right)^2 + \left(\dfrac{dF}{dz}\right)^2\right\}}} = \frac{r^2 \sin\theta\, d\theta\, d\phi}{\dfrac{dF}{dr}} \ldots.(2),$$

where x, y, z are rectangular coordinates, r, θ, ϕ polar coordinates, $F = 0$ is the equation of the surface, and dS is an element of area. It may be observed that the element (1) is proportional to the mass of the shell formed by the surfaces $F = 0$, $F + k = 0$, at the point x, y, k being indefinitely small.

Let us proceed then to investigate whether there is any relation connecting the expressions (1) at the points where the surface $F = 0$ is met by a line. Now it is evident that, in order that the elements of the double integrals should be connected by any relation, the line should involve only two parameters, or, in other words, should belong to a congruency; we may take then as its equation

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = \rho, \text{ say}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

β, γ, l, m, n are supposed to be functions of two parameters

p, q. Substituting these values (3) of x, y, z in terms of ρ in $F=0$, we get an equation of the form

$$F \equiv f(\rho) = a\rho^n + b\rho^{n-1} + \ldots = 0 \quad \text{.....................(4)},$$

to determine the n values of ρ in terms of p, q.

Now, if $du = \dfrac{dx\,dy}{\dfrac{dF}{dz}}$, we have, from (2),

$$du = \frac{l\,dy\,dz + m\,dz\,dx + n\,dx\,dy}{l\dfrac{dF}{dx} + m\dfrac{dF}{dy} + n\dfrac{dF}{dz}}.$$

But $l\dfrac{dF}{dx} + m\dfrac{dF}{dy} + n\dfrac{dF}{dz}$ is evidently equal to $\dfrac{dF}{d\rho}$, and for $dy\,dz$, &c. we substitute $\left(\dfrac{dy}{dp}\dfrac{dz}{dq} - \dfrac{dy}{dq}\dfrac{dz}{dp}\right) dp\,dq$, &c. Thus we get

$$du = \frac{G\,dp\,dq}{\dfrac{dF}{d\rho}} \quad \text{..............................(5)},$$

where
$$G = \begin{vmatrix} l, & m, & n \\ \dfrac{dx}{dp}, & \dfrac{dy}{dp}, & \dfrac{dz}{dp} \\ \dfrac{dx}{dq}, & \dfrac{dy}{dq}, & \dfrac{dz}{dq} \end{vmatrix}.$$

Now, from (3),
$$\frac{dx}{dp} = \frac{da}{dp} + \rho\frac{dl}{dp} + l\frac{d\rho}{dp},$$
$$\frac{dx}{dq} = \frac{da}{dq} + \rho\frac{dl}{dq} + l\frac{d\rho}{dq}, \quad \frac{dy}{dp} = \text{\&c.}$$

Hence, substituting these values in G, $\dfrac{d\rho}{dp}$ and $\dfrac{d\rho}{dq}$ disappear, and we may write

$$G = L\rho^2 + M\rho + N \quad \text{..............................(6)},$$

where
$$L = \begin{vmatrix} l, & m, & n \\ \dfrac{dl}{dp}, & \dfrac{dm}{dp}, & \dfrac{dn}{dp} \\ \dfrac{dl}{dq}, & \dfrac{dm}{dq}, & \dfrac{dn}{dq} \end{vmatrix}, \; M = \text{\&c.}$$

Now, if we seek the geometrical signification of $G=0$, it is easy

see that it gives the points where the line is intersected by a consecutive; that is, the two points where the line in general is bitangent to a surface. Hence, if r_1, r_2 are the distances of these points from α, β, γ, we have, from (5),

$$du = L(\rho - r_1)(\rho - r_2)\frac{dp\,dq}{f'(\rho)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7),$$

where we have put $f'(\rho)$ for $\frac{dF}{d\rho}$.

This result is exactly analogous to the corresponding theorem *in plano* (see §2 of a paper of mine in Vol. xv., No. 215, of the *Proceedings*, "On certain results obtained by means of the Arguments of Points on a Plane Curve").

Hence we see that, if lines belonging to a congruency meet a surface of the fourth order, we have

$$\Sigma_4^1 du = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

and, if they meet a surface of the fifth order, we have, besides,

$$\Sigma x du = \Sigma y du = \Sigma z du = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9),$$

and further relations for surfaces of higher orders. We may give a more exact interpretation to these results as follows. Let us take any closed area on the surface F, and through every point of this area draw lines belonging to an assigned congruency. These lines will evidently trace out $n-1$ other areas on F. Then, if u, v, &c. denote the values of $\iint du, \iint x du$, &c. over one of these areas, we have, from (8) and (9), $\Sigma u = 0$, $\Sigma v = 0$, &c.

I now proceed to apply the result (7) to a quadric. If a line belonging to a congruency meet a quadric, we have

$$du_1 = \left(\frac{L}{a}\right)\frac{AO_1 \,.\, AO_2 \,.\, dp\,dq}{AB},$$

$$du_2 = \left(\frac{L}{a}\right)\frac{BO_1 \,.\, BO_2 \,.\, dp\,dq}{BA},$$

where A, B are the points on the quadric, and O_1, O_2 are the points of contact with the surface to which the line is bitangent. Hence we obtain

$$\frac{du_1}{AO_1 \,.\, AO_2} + \frac{du_2}{BO_1 \,.\, BO_2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

Suppose the line to be a bitangent to a quartic surface U, then it is easy to show that

$$AO_1^2 \,.\, AO_2^2 \propto U_1, \quad BO_1^2 \,.\, BO_2^2 \propto U_2,$$

where U_1, U_2 are the results of substituting the coordinates of A, B, respectively, in U. Thus, in this case, (10) gives

$$\frac{du_1}{\sqrt{U_1}} \pm \frac{du_2}{\sqrt{U_2}} = 0 \quad\text{.............................(11).}$$

If we suppose the line to be an inflexional tangent of a cubic surface U, O_1 and O_2 coincide, and we can show that

$$\frac{AO_1^3}{BO_1^3} = \frac{U_1}{U_2}.$$

Hence in this case we have

$$\frac{du_1}{(U_1)^{\frac{1}{3}}} + \frac{du_2}{(U_2)^{\frac{1}{3}}} = 0 \quad\text{..........................(12).}$$

Again, suppose the line to be a chord of a curve lying on another quadric V, then, by a property of the quadric, we have

$$\frac{AO_1 \,.\, AO_2}{BO_1 \,.\, BO_2} = \frac{V_1}{V_2};$$

therefore
$$\frac{du_1}{V_1} + \frac{du_2}{V_2} = 0 \quad\text{..............................(13).}$$

Going back now to (11), it may be observed that we cannot, as *in plano*, infer the existence of a doubly infinite number of polygons inscribed in a quadric F whose sides are bitangents to quartics of the form
$$U + SF = 0 \quad\text{..............................(14),}$$
where U is a given quartic and S is a quadric which is different for each side of the polygon. The result (11), however, suggests the existence of such polygons in certain cases.

In fact, we can show independently that such polygons exist in the case where the quartics (14) coincide and break up into two quadrics inscribed in the same developable as F. We know that three quadrics inscribed in the same developable can be converted homographically so as to become confocal. Now Liouville has obtained the differential equations of the lines touching the confocal quadrics $\rho = a_1$, $\rho = a_2$, in the form

$$\left.\begin{aligned} L(\rho) \pm L(\mu) \pm L(\nu) = \alpha \\ M(\rho) \pm M(\mu) \pm M(\nu) = \beta \end{aligned}\right\} \quad\text{.....................(15),}$$

where

$$L(\rho) = \int \frac{d\rho}{\sqrt{\{(\rho^2-h^2)(\rho^2-k^2)(\rho^2-a_1^2)(\rho^2-a_2^2)\}}},$$

$$M(\rho) = \int \frac{\rho^2\, d\rho}{\sqrt{\{(\rho^2-h^2)(\rho^2-k^2)(\rho^2-a_1^2)(\rho^2-a_2^2)\}}},$$

α, β are constants, and the notation is the usual one of elliptic coordinates (Liouville's *Journal de Mathématiques*, t. xii., p. 418).

Hence, if we put

$$L(\mu) \pm L(\nu) = u, \quad M(\mu) \pm M(\nu) = v,$$

we easily find, for the two points on the surface $\rho =$ a constant, where it is met by a common tangent of the surfaces, $\rho = a_1$, $\rho = a_2$,

$$u_1 - u_2 = 2L(\rho), \quad v_1 - v_2 = 2M(\rho) \quad \ldots\ldots\ldots\ldots\ldots (16).$$

Hence, if Ω, Ω' are complete values of the integrals $L(\rho)$, $M(\rho)$, respectively, for a polygon of n sides we evidently obtain $2nL(\rho) = \Omega$, $2nM(\rho) = \Omega'$, which serve to determine two relations connecting ρ, a_1, a_2. Thus we see that, if these two conditions are satisfied, there are a doubly infinite number of closed polygons inscribed in the quadric ρ whose sides touch the quadrics a_1, a_2. It may be added that this result may be extended to the case in which the vertices of the polygon lie on different confocals.

We may now proceed to the application of (7) to the cubic surface F.

Suppose the lines of the congruency to be bitangents of the quartic surface U, then it is easy to see that, for the three systems of points on F, we have

$$\Sigma \frac{du}{\sqrt{U}} = 0, \quad \Sigma \frac{x\,du}{\sqrt{U}} = \Sigma \frac{y\,du}{\sqrt{U}} = \Sigma \frac{z\,du}{\sqrt{U}} = 0 \quad \ldots\ldots\ldots\ldots (17),$$

and, if the lines are inflexional tangents of a cubic surface U, we have

$$\Sigma \frac{du}{(U)^{\frac{2}{3}}} = 0, \quad \Sigma \frac{x\,du}{(U)^{\frac{2}{3}}} = \Sigma \frac{y\,du}{(U)^{\frac{2}{3}}} = \Sigma \frac{z\,du}{(U)^{\frac{2}{3}}} = 0 \ldots\ldots\ldots\ldots (18).$$

Also, if the lines satisfy the single condition of touching a quadric U, we have always

$$\Sigma \frac{du}{\sqrt{U}} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19).$$

Again, let one of the points C where a line of the congruency meets the surface lie on a fixed curve, then O_2, say, coincides with C, and for the other two points A, B we easily find

$$\frac{du_1}{AO_1} + \frac{du_2}{BO_1} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20).$$

Hence if, in this case, the line also touch a quadric U, we have

$$\frac{du_1}{\sqrt{U_1}} \pm \frac{du_2}{\sqrt{U_2}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21),$$

or, if it intersect a curve lying in a plane P, we evidently get

$$\frac{du_1}{P_1}+\frac{du_2}{P_2}=0 \quad \text{.........................(22).}$$

The result (21) appears to suggest certain theorems concerning the possibility of the inscription of an infinite number of closed polygons.

Proceeding now to surfaces of the fourth order, we have seen that $\Sigma du = 0$ in general. Hence, if the lines of the system intersect two fixed curves on the surface, we have for the two points where they meet the surface again,

$$du_1+du_2=0 \quad \text{...................................(23).}$$

Thus we see that $\iint du$ has the same value over two portions of area traced out on the surface by lines meeting it again in two fixed curves. It may be shown that we have also this relation (23) in the case in which the lines are bitangents to a quartic U touching the given surface F along its intersection with a quadric. We have $F = U+S^2$, identically, where S is a quadric. Now, if the line is a bitangent of U, we must have, for the points of contact, U equal to a perfect square $= Q^2$. We thus find

$$\frac{(\alpha-x)(\alpha-y)}{(\alpha-\gamma)(\alpha-\delta)}-\frac{(\beta-x)(\beta-y)}{(\beta-\gamma)(\beta-\delta)}=0 \quad \text{..............(24),}$$

where x, y give the points of contact with U, and α, β, γ, δ the points of intersection with F. Hence, from (7), we obtain the relation (23).

We may extend the preceding results to the case of a variable conic intersecting a fixed surface. An arbitrary conic can evidently be expressed by means of the equations

$$x=\frac{f_1}{f_4},\quad y=\frac{f_2}{f_4},\quad z=\frac{f_3}{f_4},$$

where $$f_1=a_1t^2+b_1t+c_1,\quad f_2=\text{\&c.}$$

We suppose then a_1, b_1, c_1, &c. to be functions of two parameters p, q, and we find thus, as in the case of the line,

$$du=\frac{G\,dp\,dq}{f'(t)} \quad \text{.................................(25),}$$

where G is now a polynomial of the sixth degree in t, whose vanishing gives the values of t corresponding to the points where the conic is touched by a surface. Hence we obtain the equations (8) and (9), the summation extending to all the points of intersection of the conic and the surface.

In exactly the same way we can find relations between the elements of the double integrals corresponding to the points where a surface is intersected by a unicursal curve of any degree.

In the same way as in the paper of mine referred to above, we can arrive at several relations connecting the points of contact of the sides of triangles inscribed in a surface whose sides belong to given congruences. For instance, from (10), we find that, if there be a system of triangles inscribed in a quadric whose sides belong to congruences, then the six points of contact will lie on a conic.

Again, if there be a system of triangles inscribed in a cubic surface whose sides meet the surface again on fixed curves, then, if the sides are also tangents to surfaces, the three points of contact will lie on a line. Also, if lines belonging to congruences form triangles inscribed in a cubic so that the points where the sides meet the surface again lie on a line, then the six points of contact will lie on a conic. Several other results can be deduced from (10). Since du for a sphere is an element of area, it can be shown that, if tangents be drawn to a sphere so as to intersect a fixed curve at infinity, these lines will trace out on a concentric sphere portions of equal area.

Again, we can find a relation between the areas cut off from the cyclide

$$U \equiv (x^2+y^2+z^2+k^2)^2-4\,(a^2x^2+b^2y^2+c^2z^2) = 0$$

by lines varying in a certain manner. For this surface we easily find

$$dS = \sqrt{\{a^2\,(a^2-k^2)\,x^2+b^2\,(b^2-k^2)\,y^2+c^2\,(c^2-k^2)\,z^2\}}\,du,$$

where dS is an element of area. If then we take lines bitangent to the surface $U+\lambda F = 0$, where

$$F \equiv a^2\,(a^2-k^2)\,x^2+b^2\,(b^2-k^2)\,y^2+c^2\,(c^2-k^2)\,z^2,$$

we have
$$\frac{du}{\sqrt{F}} \propto \frac{1}{f'(\rho)}, \text{ from (7).}$$

Hence, since
$$\Sigma\frac{x^2\,du}{\sqrt{F}} = \Sigma\frac{y^2\,du}{\sqrt{F}} = \Sigma\frac{z^2\,du}{\sqrt{F}} = 0,$$

we have $\Sigma\sqrt{F}\,du = \Sigma dS = 0$. Thus we see that a series of bitangents to the surface $U+\lambda F = 0$ will trace out on the given surface four areas which are connected by the relation $S_1 \pm S_2 \pm S_3 \pm S_4 = 0$; and it is evident that the boundary of one of these areas can be assumed arbitrarily.

On some Properties of certain Solutions of a Differential Equation of the Second Order. By Dr. E. J. Routh.

[*Read April 2nd,* 1885.]

If we write down Legendre's equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)\,y=0 \;\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

we have a solution, when n is a positive integer, usually called a Legendre's coefficient, and written $y=P_n$. The chief properties of this function may be briefly summed up. (1) It is an n^{th} differential coefficient. (2) It satisfies a scale of relation by which P_{n+2} is found in terms of P_n and P_{n+1}. (3) All the roots of the equation $P_n=0$ are real, and are confined between certain limits. (4) The integral $\int P_m P_n\,dx$ vanishes between the same limits, while $\int P_n^2\,dx$ has a known value. Also (5) P_n may be generated as the coefficient of t^n by the expansion of a known function of t.

Let us now generalize the differential equation by introducing arbitrary letters into every coefficient. It now becomes

$$(ax^2+bx+c)\frac{d^2y}{dx^2}+(fx+g)\frac{dy}{dx}+hy=0 \;\ldots\ldots\ldots\ldots\ldots(2),$$

where h is a function of some parameter n, which we shall afterwards take to be a positive integer. Two solutions at least of this generalized equation have been given which are n^{th} differential coefficients, each of which reduces to P_n when the differential equation takes the form (1). Representing either of these by $y=X_n$, we propose to show that the corresponding scale of relation is of the form

$$AX_{n+2}+(B+Cx)\,X_{n+1}+DX_n=0,$$

where (A, B, C, D) are constants which depend on (a, b, c, f, g, n). Following the same general line of argument as that used in the case of Legendre's coefficients, we shall show that, provided $4-f/a$ in the first solution or f/a in the second is positive, and in each case numerically greater than $\dfrac{2g-bf/a}{\pm\,(b^2-4ac)^{\frac{1}{2}}}$ (the root being supposed real), the properties of X_n/X_0 follow exactly those just mentioned for P_n.

The differential equation (2) has been discussed by a great many authors, chiefly with the object of finding its general integral. For some of the following references the writer is indebted to the kindness of the referees of this paper.

An equation which reduces to the form (2) by easy transformations is discussed by Euler in his Integral Calculus. He obtains in certain cases solutions in which y is expressed in a finite series of powers of x. As he does not put these into the form of an n^{th} differential coefficient, his solutions do not at present concern us. In 1812 or 1813, Gauss (Collected Works, Vol. III.) expressed the solution as a hypergeometrical series. Liouville, in the thirteenth volume of the *Journal de l'Ecole Polytechnique*, 1832, discusses the equation in a manner which, in the beginning, is the same as that in Art. 3 of this paper. He obtains an integral in the form of an n^{th} differential coefficient, and discusses the meaning of the result when n is not an integer. In 1860, Spitzer treats of that particular case of the equation (2) in which $a=0$, and expresses the solutions as n^{th} differential coefficients. Then, in 1868, H. J. Holmgren (Kongliga Svenska Vetenskaps-Akademiens Handlingar), after giving references to these works of Euler, Liouville, and Spitzer, mentions certain restrictions on their solutions. He takes Liouville's paper as his point of departure, and finds integrals in which these restrictions are removed. The integrals are expressed as n^{th} differentials, and the meaning when n is not a positive integer is given. The solution of the differential equation (2) is also discussed by Boole, under the name of Pfaff's equation, in the latter part of his Differential Equations. In Pfaff's treatise we find a great many cases specified in which the equation has been solved both as n^{th} differential coefficients or otherwise. Probably no case in which n is integral has escaped his search.

After so many writers have obtained solutions, it is not to be supposed that anything new is presented in the two n^{th} differentials found in Arts. 3 and 4. Taking these two forms as the point of departure, we pass on to the scale of relation as the theorem next in order. None of the writers already mentioned seem to have alluded to this result, so that it is possibly new.

Taking next the theorems on the roots of $X_n=0$, we find two papers by Sturm in the first volume of Liouville's *Journal*. In the first of these, he discusses the very general equation

$$\frac{d}{dx}\left(k\frac{dy}{dx}\right)+gy=0,$$

where k and g are functions of x and some parameter n. Subject to certain specified conditions, he proves theorems concerning the separation of the roots. Afterwards he discusses a theorem equivalent to $\int X_m X_n \phi(x)\, dx = 0$. His reasoning is necessarily very different from that given in this paper, for here the scale of relation is the central proposition with which all the others are connected, and

this theorem is not alluded to in Sturm's papers. Even therefore, on this point of contact with Sturm's paper, what is here stated presents some novelty.

The value of $\int \phi(x) X_n^2\, dx$ has been found between the limits which make $ax^2+bx+c=0$.

The generating function of X_n is also found, but it seems too complicated to be of very great use.

Finally, two special cases, in which exponentials occur in the value of y, are alluded to at the end of the paper.

1. The two solutions of the differential equation

$$(ax^2+bx+c)\frac{d^2y}{dx^2}+(fx+g)\frac{dy}{dx}+hy=0 \quad \text{..............(2)},$$

which are the subjects of consideration, may be found by two artifices. *First*, we may put $y=\dfrac{d^{n+1}z}{dx^{n+1}}$; then, if the equation be a perfect n^{th} differential, the value of z can be at once found. *Secondly*, we may seek some factor μ by which the equation (2) can be made a perfect differential.

2. By writing $px+q$ for x, we can always reduce the equation to the form

$$(1-x^2)\frac{d^2y}{dx^2}+(f'x+g')\frac{dy}{dx}+h'y=0 \quad \text{.................(3)}.$$

This form is sometimes more convenient than (2). But, when the factors of ax^2+bx+c are not real, the values of p and q are not real. To avoid these imaginary quantities, we shall use the general form in obtaining the solutions.

3. *The First Solution.*—Writing $y=d^{n+1}z\,/\,dx^{n+1}$, and integrating each term $(n+1)$ times by the help of Leibnitz's theorem for negative indices; viz.

$$\delta^{-n}uv=v\,\delta^{-n}u-n\frac{dv}{dx}\delta^{-n-1}u+n\frac{n+1}{2}\frac{d^2v}{dx^2}\delta^{-n-2}u-\&\text{c.},$$

where δ stands for $d\,/\,dx$; we have

$$(ax^2+bx+c)\frac{d^2z}{dx^2}+\{fx+g-(n+1)(2ax+b)\}\frac{dz}{dx}+h'z=0,$$

where $h'=h-(n+1)f+(n+1)(n+2)\,a$, and the constants of integration have been omitted.

If n be so chosen that $h'=0$, we find, by an easy integration,

$$\frac{dz}{dx}=(ax^2+bx+c)^{n+1}\frac{1}{R},$$

where $$\frac{dR}{dx}=\frac{fx+g}{ax^2+bx+c}R.$$

Summing up, we may state the theorem thus. Let α and β be the roots of the quadratic

$$(n+1)(n+2)a-(n+1)f+h=0 \quad\text{.....................(4).}$$

Then a solution of the differential equation (2) is

$$y=A\frac{d^\alpha}{dx^\alpha}\frac{(ax^2+bx+c)^{\alpha+1}}{R}+B\frac{d^\beta}{dx^\beta}\frac{(ax^2+bx+c)^{\beta+1}}{R},$$

where $$\frac{1}{R}\frac{dR}{dx}=\frac{fx+g}{ax^2+bx+c} \quad\text{.........................(5).}$$

If either α or β is a positive integer, a solution of the required form has been found.

4. *The Second Solution.*—Let us now multiply the differential equation (2) by an integrating factor μ. The condition of integrability leads to the equation

$$h\mu-\frac{d}{dx}(fx+g)\mu+\frac{d^2}{dx^2}(ax^2+bx+c)\mu=0.$$

Expanding, this takes the form

$$(ax^2+bx+c)\frac{d^2\mu}{dx^2}+\{(4a-f)x+2b-g\}\frac{d\mu}{dx}+(h-f+2a)\mu=0.$$

The integral of the given equation is then

$$(ax^2+bx+c)\mu\frac{dy}{dx}+\left\{(fx+g)\mu-\frac{d}{dx}(ax^2+bx+c)\mu\right\}y=C,$$

where C is some constant. Choosing C to be zero, as the integral can be found without its help, we have

$$y=\frac{ax^2+bx+c}{R}\mu,$$

where R has the same meaning as before.

The equation to find μ is of the form already discussed in the first solution. A solution of the required form can be found if either of

the values of m given by the quadratic

$$(m+1)(m+2)a-(m+1)(4a-f)+h-f+2a=0$$

is an integer. This quadratic can be written in the form

$$m(m-1)a+mf+h=0.$$

Summing up, we may state the theorem thus. Let α and β be the roots of the quadratic

$$m(m-1)a+mf+h=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

then a solution of the differential equation (2) is

$$y=\frac{ax^2+bx+c}{R}\left\{A\frac{d^\alpha}{dx^\alpha}(ax^2+bx+c)^{\alpha-1}R+B\frac{d^\beta}{dx^\beta}(ax^2+bx+c)^{\beta-1}R\right\},$$

where R has the same form as before.

If either* of the roots of this quadratic is an integer, a solution of the differential equation as an n^{th} differential coefficient has been found.

5. Comparing the two solutions, we now see that we could derive one from the other (say, the second from the first) by a simple substitution. For, if we write $y=\mu(ax^2+bx+c)/R$, the equation to find μ is of the form solved in the first solution, with $4a-f$ written for f and $2b-g$ for g.

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* If both the roots of either of the fundamental quadratics are positive integers, it might be supposed that each root would give a solution in the form of an n^{th} differential. But the two solutions thus found are really the same. To show this, we shall use the following theorem, which has been obtained by an application of Leibnitz's theorem. If $2k+1$ be an integer, though k may be a fraction,

$$\frac{d^{2k+1}}{dx^{2k+1}}(x^2-1)^k\left(\frac{x-1}{x+1}\right)^l=2^{2k+1}M(x^2-1)^{-k-1}\left(\frac{x-1}{x+1}\right)^l,$$

where $\qquad M=(k+l)(k+l-1)\ldots$ to $2k+1$ factors.

To shorten the algebraical processes, let us take the differential equation in its simplified form (3). The second solution, after substitution for R, becomes

$$y\frac{R}{1-x^2}=A\left(\frac{d}{dx}\right)^\alpha(1-x^2)^{\frac{1}{2}(\alpha-\beta-1)}Q^{\frac{1}{2}g}+B\left(\frac{d}{dx}\right)^\beta(1-x^2)^{\frac{1}{2}(\beta-\alpha-1)}Q^{\frac{1}{2}g},$$

where $Q=\dfrac{1+x}{1-x}$.

But, by the theorem just mentioned,

$$\left(\frac{d}{dx}\right)^{\alpha-\beta}(x^2-1)^{\frac{1}{2}(\alpha-\beta-1)}Q^{\frac{1}{2}g}=2^{\alpha-\beta}M(x^2-1)^{\frac{1}{2}(\beta-\alpha-1)}Q^{\frac{1}{2}g}.$$

Differentiating this β times, we see that the two integrals are the same, or (since M may vanish) one of them is zero identically. The other solution may be treated in the same way.

where α, β, γ are three functions of the coefficients A, B, C, D. Equating these three to zero, and (to avoid fractions as much as possible) choosing the value of A as indicated above, we find the values of B, C, D.

11. If we substitute $(x^2-1)^q Q$ for Q, we may easily find a useful extension of this theorem. We then have

$$\left.\begin{aligned} X_n &= \frac{d^n}{dx^n}(x^2-1)^{n+q} Q \\ (x^2-1)\frac{dQ}{dx} &= \{(M-2q)x+N\} Q \end{aligned}\right\},$$

and it follows, as before, that

$$AX_{n+2}+(B+Cx)X_{n+1}+DX_n = 0,$$

where A, B, C, D have the same values as before, and q is any integer, positive or negative.

12. If we substitute $(2ax+b)/(b^2-4ac)^{\frac{1}{2}}$ for x, and make some slight changes in the constants in order to simplify the result, we have the following theorem.

Let
$$X_n = \frac{d^n}{dx^n}\left(\frac{ax^2+bx+c}{a}\right)^{n+q} Q,$$

$$(ax^2+bx+c)\frac{dQ}{dx} = \left\{(M-2q)\frac{2ax+b}{2}+N\right\} Q,$$

then will
$$AX_{n+2}+(B+Cx)X_{n+1}+DX_n = 0,$$

where $A = -(M+n+2)(M+2n+2)$,

$$B = M(M+2n+3)\frac{N}{a}+\frac{b}{2a}C,$$

$$C = (M+2n+2)(M+2n+3)(M+2n+4),$$

$$D = (n+1)(M+2n+4)\left\{\left(\frac{N}{a}\right)^2-(M+2n+2)^2\frac{b^2-4ac}{4a^2}\right\}.$$

In the *first solution*, viz., that of Art. 3, we have $q=1$, $Q=1/R$,

so that
$$-\frac{1}{Q}\frac{dQ}{dx} = \frac{1}{R}\frac{dR}{dx} = \frac{fx+g}{ax^2+bx+c},$$

therefore
$$M = -\frac{f}{a}+2, \quad N = \frac{bf}{2a}-g.$$

In the *second solution*, viz., that of Art. 4, we have $q=-1$, $Q=R$, so that

$$M=\frac{f}{a}-2,\quad N=-\frac{bf}{2a}+g.$$

Substituting these values of M and N, we have the scale of relation for each solution.

13. Let us apply the scale of relation to analyse the first solution, which we may conveniently write in the form

$$y=\frac{d^n}{dx^n}\left(\frac{ax^2+bx+c}{a}\right)^{n+1}\frac{1}{R}=X_n.$$

When $n=0$, $\quad X_0=\dfrac{ax^2+bx+c}{aR}$;

when $n=1$, $\quad X_1=\dfrac{ax^2+bx+c}{aR}\left\{\left(4-\dfrac{f}{a}\right)x+\dfrac{2b-g}{a}\right\}.$

Thus it appears that X_0 and X_1 have a common factor; hence, by the scale of relation, all the succeeding functions have the same factor. Also, since x enters into the scale of relation only in one coefficient, and in the first power, we see that the general term must be of the form

$$X_n=\frac{ax^2+bx+c}{R}\{G_nx^n+G_{n-1}x^{n-1}+\ldots+G_1x+G_0\},$$

where G_0, G_1, &c. are all constants. We may conveniently write this solution in the form

$$y=\frac{ax^2+bx+c}{R}G_n(x).$$

14. The scale of relation will fail to give X_{n+2} when X_{n+1} and X_n are known, if either $M+n+2=0$ or $M+2n+2=0$. Since $M=-f/a+2$, this would require either n or $2n$ to be equal to $f/a-4$. But it will be immediately seen that we shall take $4-f/a$ to be positive, so this case will not occur.

We may also notice that X_n/X_0 might be of less dimensions than n if the quantity represented by C in the scale of relation could vanish. But this cannot occur if $4-f/a$ is positive.

It follows also from the same supposition that A and C must always have opposite signs; thus the coefficients of the highest powers of x in the series 1, X_1/X_0, X_2/X_0, &c., are all of the same sign, and therefore positive.

No two consecutive functions in this series can vanish for the same

value of x; for, if so, every function would vanish. This cannot occur, because the first function in the series is unity.

15. The constants A and D in the scale of relation will have the same sign if their product is positive. Remembering that

$$M = -f/a+2, \quad N = bf/2a-g,$$

we see that this will happen for all positive values of n, provided—

(1) $(-f/a+4)$ is positive, greater than zero, and numerically greater than $\dfrac{bf/a-2g}{(b^2-4ac)^{\frac{1}{2}}}$;

(2) The factors of ax^2+bx+c are real.

If these two conditions are satisfied, the series of functions 1, X_1/X_0, &c., are such that, when any one of them, as X_{n+1}/X_0, vanishes the two on each side, viz., X_n/X_0 and X_{n+2}/X_0, have opposite signs. They therefore resemble Sturm's functions, and may be used like those functions.

If we continue the proof as in Sturm's theorem, using the results of the last article, we see that the roots of the equations

$$\frac{X_1}{X_0} = 0, \quad \frac{X_2}{X_0} = 0, \quad \frac{X_3}{X_0} = 0, \quad \&c.,$$

are all real, and that the roots of each separate or lie between the roots of the next in order.

16. The necessary conditions that this should happen are that A and D should have the same sign when $n=0$, and that $4-f/a$ should be positive. The first of these conditions implies that X_2/X_0 should be negative when $X_1/X_0 = 0$. We therefore infer, *conversely, that if* $4-f/a$ *be positive, the one root of* $X_1/X_0 = 0$ *cannot lie between those of* $X_2/X_0 = 0$ *unless* $4-f/a$ *be also numerically greater than*

$$\frac{bf/a-2g}{(b^2-4ac)^{\frac{1}{2}}}.$$

17. Again, by referring to Art. 8, we see that both conditions may be expressed by making the exponents p and q of the two factors of R algebraically less than 2. Now, when this happens, the expression $\dfrac{(ax^2+bx+c)^{n+1}}{R}$ is finite for all values of x when n is any positive integer except zero. It therefore follows that the roots of all its differential coefficients up to the n^{th} inclusive, *i.e.*, the *roots of* $X_n = 0$, *are not only real, but lie between the roots of* $ax^2+bx+c = 0$.

18. Let us apply the scale of relation to analyse the second solution (Art. 4), which we may write

$$y = \frac{ax^2+bx+c}{aR}\,\frac{d^n}{dx^n}\left(\frac{ax^2+bx+c}{a}\right)^{n-1} R = X_n.$$

When $n = 0$, $\qquad X_0 = 1$,

$n = 1$, $\qquad X_1 = \dfrac{fx+g}{a}$.

By continual substitution in the scale of relation, we find X_2, &c. It appears at once that the general form of the solution is given by

$$X_n = H_n x^n + H_{n-1} x^{n-1} + \ldots + H_0,$$

where the H's are all constants.

The same remarks apply here as in the first solution, Art. 14. Since $M = f/a - 2$, it follows that A and C cannot vanish, provided f/a is positive. Thus X_n is an integral rational function of n* dimensions, and the coefficient of the highest power is positive.

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* Conversely, we may enquire what the condition is that an integral solution should exist. Taking the differential equation in its simplified form (Art. 2), viz.,

$$(1-x^2)\frac{d^2y}{dx^2} + (fx+g)\frac{dy}{dx} + hy = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

let us suppose such a solution to be

$$y = A_m x^m + A_{m-1}x^{m-1} + \ldots + A_p x^p + \ldots \quad \ldots\ldots\ldots\ldots(\text{I.}).$$

Substituting, and taking only the highest powers of x which enter, we have

$$m(m-1-f) - h = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{II.}).$$

This quadratic is the same as that used in the second solution when $a = -1$ (Art. 4). We see that there can be no highest power in the series unless this quadratic has a positive integral root. Thus the differential equation has no solution in integral positive powers of x except in those cases in which we may apply the method given in the text.

The determination of this integral solution is not further connected with the subject of this paper. But one or two points may be noticed in passing. Let α and β be the roots of the quadratic (II.), and let $m = \alpha$ be the root, supposed integral, which we take as the highest power of x. Returning to the substitution from (I.) in (3), we find

$$-(\alpha-1-\beta)A_{\alpha-1} = g\alpha A_\alpha,\quad (p-\alpha)(p-\beta)A_p = g(p+1)A_{p+1} + (p+1)(p+2)A_{p+2} \quad \ldots\ldots\ldots(\text{III.}).$$

Thus each coefficient of the series for y is found in terms of the preceding ones. Now $A_{-1} = 0$ because its $p+1 = 0$, and $A_{-2} = 0$ because its $p+2 = 0$ and $A_{-1} = 0$. The series found therefore terminates before any negative powers of x are introduced. It is clear that none of these values of A_p can be infinite unless the quadratic (II.) has two integral positive roots, and then only if $\alpha > \beta$. Even in this case, if it should happen that

$$gA_{\alpha+1} + (\alpha+2)A_{\alpha+2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(\text{IV.}),$$

19. The constants A and D in the scale of relation will have the same sign if their product is positive. This will be the case provided—

(1) f/a is positive, greater than zero, and numerically greater than $\dfrac{bf/a-2g}{(b^2-4ac)^{\frac{1}{2}}}$;

(2) The factors of ax^2+bx+c are real.

Following the same reasoning as before, we see that the functions X_0, X_1, X_2, &c. resemble Sturm's functions. *The roots of each are therefore all real and separate, or lie between the roots of the function next in order.*

The conditions may also be expressed by making both the exponents (viz., p and q) of the value of R given in Art. 8 to be positive. From this we infer, as in Art. 16, that the roots of $X_n = 0$ *are not only real, but lie between the roots of* $ax^2+bx+c = 0$.

These theorems concerning the second solution may be deduced from those for the first by using the transformation given in Art. 4.

On the value of $\int \phi(x)\, X_m X_n\, dx$.

20. We next propose to examine under what circumstances the equation $\int \phi(x)\, X_m X_n dx = 0$ may be true, and to find the value of $\int \phi(x)\, X_n^2\, dx$, where $\phi(x)$ is some known function of x.

The differential equation considered is

$$(ax^2+bx+c)\frac{d^2y}{dx^2}+(fx+g)\frac{dy}{dx}+hy = 0,$$

---

then A_a becomes arbitrary instead of infinite. Putting $A_a = 0$, or giving it any value we please, we then find A_{a-1}, $A_{a-2} \dots A_0$ by the formulæ (IV.). When this condition happens to be satisfied, the differential equation (3) has two solutions in positive integral powers of x.

As an example, consider the differential equation

$$(1-x^2)\frac{d^2y}{dx^2}+(fx+g)\frac{dy}{dx}+2(1-f)\,y = 0.$$

Here the quadratic leads to $m=2$ or $m=f-1$; choosing the former, we find, without difficulty, $y = A\,[(f-1)(f-2)\,x^2+2g\,(f-1)\,x+f-2+g^2]$.

If $g=0$ and $f=2$, this value of y becomes $y = B\,(x^2+1)$ or $y = Cx$, according as we absorb the $f-2$ or the g into the arbitrary multiplier. If $g=0$ and $f=4$, the value of y becomes $y = 2A\,(3x^2+1)$. All these are easily seen to be solutions.

Holmgren (to whose paper the writer had a reference) has shown that the equation (3) admits of two integral solutions when (in our notation) α, β, and $\frac{1}{2}(\alpha-\beta-1\pm g)$ are all positive integers, α being $> \beta$. In this case, the solution derived from the greater root may be written as an n^{th} differential in the form

$$y = (x+1)^{\frac{1}{2}(\alpha+\beta-g+1)}\left(\frac{d}{dx}\right)^{\frac{1}{2}(\alpha-\beta-g-1)}\left[\frac{(x-1)^{\alpha}}{(x+1)^{\beta+1}}\right].$$

where h is a function of n. Writing m for n, let z be the corresponding value of y. We then have

$$(ax^2+bx+c)\frac{d^2z}{dx^2}+(fx+g)\frac{dz}{dx}+h'z=0.$$

We easily find (as Laplace does in treating of his functions) that

$$(h'-h)\,yz=(ax^2+bx+c)\left(y\frac{d^2z}{dx^2}-z\frac{d^2y}{dx^2}\right)+(fx+g)\left(y\frac{dz}{dx}-z\frac{dy}{dx}\right).$$

We now multiply both sides by the integrating factor of the right-hand side. This factor is easily seen to be $\frac{R}{ax^2+bx+c}$, where R has the same value as before (Art. 8). We then have

$$(h'-h)\int yz\,\frac{R\,dx}{ax^2+bx+c}=R\left(y\frac{dz}{dx}-z\frac{dy}{dx}\right),$$

where both sides are to be taken between the same limits.

21. If we adopt as our values of y and z those given by the first solution as analysed in Art. 13, we write for y and z their values

$$y=\frac{ax^2+bx+c}{R}\,G_n(x),\quad z=\frac{ax^2+bx+c}{R}\,G_m(x).$$

The right-hand side of the equation will now become

$$\frac{(ax^2+bx+c)^2}{R}\left\{G_n(x)\frac{dG_m(x)}{dx}-G_m(x)\frac{dG_n(x)}{dx}\right\}.$$

Since the G-functions are all integral and rational when m and n are positive integers, we see that this will vanish when the roots of $ax^2+bx+c=0$ are real, and the limits are chosen to be those roots. It is, however, also necessary that the exponents of the two factors of R should not be so large as those of the corresponding factors in $(ax^2+bx+c)^2$. This requires that each of the exponents, viz., p and q, which occur in R (see Art. 8), should be algebraically less than 2. These conditions are the same as those given in Art. 17 as the conditions that the roots of the function $G(x)=0$ should all be real.

The proof requires that h and h' should not be equal except when $n=m$. Referring to Art. 3, we see that in the first solution

$$h=(n+1)\{f-(n+2)\,a\}.$$

Writing m for n to obtain h', it follows that $h'=h$ only when $n=m$

and $n+m=f/a-3$. But this latter case cannot occur for any unequal positive integral values of m and n, because we have already assumed in Art. 15 that $4-f/a$ is positive and greater than zero. A similar remark applies when we use the second solution in the next article.

22. If we adopt as our values of y and z those given by the second solution as analysed in Art. 18, we see that the right-hand side of the equation arrived at in Art. 20 becomes

$$=R\left(X_n\frac{dX_m}{dx}-X_m\frac{dX_n}{dx}\right).$$

Now the functions X_m and X_n are integral rational functions of x when m and n are positive integers. Hence the right-hand side is zero if the roots of $ax^2+bx+c=0$ are real and those roots are taken as the limits. It is necessary that the exponents of the corresponding factors in the value of R should be positive. This will be the case whenever the conditions that the roots of $X_n=0$ should be all real (given in Art. 19) are satisfied.

23. *Summing up*, we see that, if y_m and y_n be two values of y corresponding to two positive integral values of the parameter in either the first or the second solution (Arts. 3 and 4), then

$$\int y_m y_n \frac{R\,dx}{ax^2+bx+c}=0,$$

if the limits are the roots of $ax^2+bx+c=0$, provided those roots are real. It is also necessary that the indices of the factors of R should be such that, when we use the first solution, $\dfrac{(ax^2+bx+c)^2}{R}$ vanishes at each limit; and, when we use the second solution, R vanishes at each limit.*

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* We have seen that the conditions necessary that $\int\phi(x)\,X_mX_n\,dx=0$ are the same as those that all the functions X_n should have their roots real. The reason for this identity will be better seen by the help of the following theorem, which may be used for functions more general than those to which it is here applied.

Theorem.—*If X_n be an integral rational function of x of n dimensions, such that $\int\phi(x)\,X_nX_m\,dx=0$ for all positive integral values of m less than n, then the roots of the equation $X_n=0$ are all real and lie between the limits of integration.* It is supposed that the limits of integration are fixed, and that $\phi(x)$ is finite and keeps one sign between those limits.

For, if possible, let $X_n=0$ have a pair of imaginary roots, or a real root which does not lie between the limits of integration. In either case, X_n has a factor which keeps one sign between the limits. Let $f(x)$ be the remaining factors, then it is obvious that $f(x)$ can be expanded in a series

$$f(x)=a_0X_0+a_1X_1+\ldots+a_{n-1}X_{n-1}.$$

24. *To find by a general method the value of* $\int \phi(x) X_n^2 dx$ *for any limits which make* $\int \phi(x) X_m X_n dx = 0$.

We have, by the scale of relation as given in Arts. 10 and 12,

$$A_n X_{n+2} + (B_n + C_n x) X_{n+1} + D_n X_n = 0,$$

where A_n, B_n, C_n, D_n are functions of n, but not of x. Multiply this equation by $X_n \phi(x)$ and integrate; we find

$$C_n \int x \phi(x) X_n X_{n+1} dx + D_n \int \phi(x) X_n^2 dx = 0.$$

Now write $(n-1)$ for n in the scale of relation, and multiply by $\phi(x) X_{n+1}$. We find, after integration,

$$A_{n-1} \int \phi(x) X_{n+1}^2 dx + C_{n-1} \int x \phi(x) X_n X_{n+1} dx = 0.$$

We immediately deduce

$$\frac{C_n}{D_n} \int \phi(x) X_{n+1}^2 dx = \frac{C_{n-1}}{A_{n-1}} \int \phi(x) X_n^2 dx.$$

From this, by continued reduction, we have

$$\int \phi(x) X_n^2 dx = \frac{C_0}{C_{n-1}} \cdot \frac{D_{n-1} D_{n-2} \dots D_1}{A_{n-2} A_{n-3} \dots A_0} \int \phi(x) X_1^2 dx.$$

25. Taking the second solution first, the value of $\int \phi(x) X_1^2 dx$ may be found in gamma integrals. Thus, putting $P = ax^2 + bx + c$, we have, by Art. 18,

$$\int \phi(x) X_1^2 dx = \int \frac{R}{P} \left(\frac{fx+g}{a}\right)^2 dx = \int \frac{fx+g}{a^2} dR,$$

by the definition of R given in Art. 8. Now, substituting for R from that article and remembering that p and q are positive (Art. 22), we get, after an integration by parts,

$$\int_\mu^\lambda \phi(x) X_1^2 dx = -\frac{f}{a^2} (\lambda - \mu)^{1+f/a} \int_0^1 (z-1)^p z^q dz.$$

In the same way we find for the first solution

$$\int_\mu^\lambda \phi(x) X_1^2 dx = -\frac{4a-f}{a^2} (\lambda - \mu)^{5-f/a} \int_0^1 (z-1)^{2-p} z^{2-q} dz.$$

Here (as in Art. 21) we have supposed that p and q are less than 2.

Now the product $\phi(x) f(x) X_n$ keeps one sign between the limits; hence every term of the integral $\int \phi(x) f(x) X_n dx$ has the same sign. But, by hypothesis, if we substitute this series for $f(x)$ we see that the integral vanishes, which is impossible.

The Generating Function.

26. We have, by Lagrange's theorem in the differential calculus,

$$u = x + t\phi(u),$$

$$f(u) = f(x) + \&c. + \frac{t^{n+1}}{L(n+1)}\frac{d^n}{dx^n}(\phi x)^{n+1} f'(x) + \&c.$$

Differentiating both sides with regard to t and substituting for du/dt, we have

$$\frac{f'(u)\,\phi(u)}{1-t\phi'(u)} = \&c. + \frac{t^n}{Ln}\frac{d^n}{dx^n}(\phi x)^{n+1} f'(x) + \&c.$$

In the *first solution* we put $f'(x) = 1/R$ and $a\phi(u) = au^2 + bu + c$. Hence, if $R(u)$ be the same function of u that R is of x in Art. 8, we have

$$\frac{au^2+bu+c}{R(u)}\,\frac{1}{a-t(2au+b)} = \&c. + \frac{t^n}{Ln}\frac{d^n}{dx^n}\left(\frac{ax^2+bx+c}{a}\right)^{n+1}\frac{1}{R} + \&c.$$

In the *second solution* we give $\phi(u)$ the same meaning as before, and put $f'(x) = R(ax^2+bx+c)^{-2}$. We find

$$\frac{R(u)}{au^2+bu+c}\,\frac{1}{a-t(2au+b)} = \&c. + \frac{t^n}{Ln}\frac{d^n}{dx^n}\left(\frac{ax^2+bx+c}{a}\right)^{n-1} R + \&c.$$

The coefficients of $\dfrac{t^n}{Ln}$ on the right-hand sides of these equations are the values of X_n given in Arts. 13 and 18 for the first and second solutions. We must, however, not forget to multiply both sides of the equation by the factor $\dfrac{ax^2+bx+c}{aR}$ in the case of the second solution. The left-hand sides of these equations are therefore known functions of t which by expansion give the two values of X_n.

27. If we substitute for u its value given by the quadratic at the beginning of this article, we have an expression for the generating function which is free from all integrations, and which takes different forms when we substitute for $R(u)$ the various forms given in Art. 8.

To take a simple illustration, let the equation be given in the form (Art. 2)

$$(1-x^2)\frac{d^2y}{dx^2} + (fx+g)\frac{dy}{dx} + hy = 0,$$

and let us adopt the second solution, $y = X_n$. Then X_n is the coefficient of t^n / Ln in the expansion of

$$\left[\frac{(1-4xt+4t^2)^{\frac{1}{2}} - (1-2xt)}{2t^2(1-x^2)}\right]^{-\frac{1}{2}f-1}\left[\frac{x-2t+(1-4xt+4t^2)^{\frac{1}{2}}}{1+x}\right]^{\frac{1}{2}g}\frac{1}{(1-4xt+4t^2)^{\frac{1}{2}}}.$$

Special Cases.

28. *The special case* in which $a = 0$ presents some interesting forms, partly because the expression for R now contains exponentials, and partly because the values of M and N in the scale of relation (Art. 12) are both infinite. The differential equation is

$$(bx+c)\frac{d^2y}{dx^2}+(fx+g)\frac{dy}{dx}+hy=0.$$

Here $$R=(bx+c)^p e^{fx/b}, \text{ where } p=\frac{bg-cf}{b^2}.$$

We shall suppose that f and b are not zero.

In the *first solution* (Art. 3) we have $h=(n+1)f$,

$$y=\frac{d^n}{dx^n}\frac{(bx+c)^{n+1}}{R}=e^{-fx/b}\left(-\frac{f}{b}+\frac{d}{dx}\right)^n(bx+c)^{n+1-p}.$$

Putting X_n for this value of y, the scale of relation is

$$X_{n+2}+\{fx+g-2(n+2)b\}X_{n+1}+b^2(n+1)(n+2-p)X_n=0.$$

In the *second solution* (Art. 4) we have $h=-nf$,

$$y=\frac{bx+c}{R}\frac{d^n}{dx^n}(bx+c)^{n-1}R=(bx+c)^{1-p}\left(\frac{f}{b}+\frac{d}{dx}\right)^n(bx+c)^{n-1+p}.$$

Putting X_n for this value of y, the scale of relation is

$$X_{n+2}-\{fx+g+2(n+1)b\}X_{n+1}+b^2(n+1)(n+p)X_n=0.$$

From these scales of relation the other properties follow without difficulty; provided in the first case $p<2$, and in the second case p is positive.

These scales of relation may be deduced from the general form given in Art. 12 by expanding the coefficients in powers of f/a. They may be obtained independently by the process explained in Art. 10.

29. Another interesting special case occurs when $a=0$, $b=0$, $c=1$. Taking the first solution, we have

$$R=e^{\frac{1}{2}fx^2+gx},\quad y=\frac{d^n}{dx^n}e^{-\frac{1}{2}fx^2-gx}=X_n.$$

The scale of relation is

$$X_{n+2}+(fx+g)X_{n+1}+(n+1)fX_n=0.$$

Theorems similar to those proved in the general case follow from this scale of relation.

Other special cases occur in which the function R has a peculiar form,—such, for instance, as that in which $a=1$, $b=0$, $c=0$.

May 14th, 1885.

J. W. L. GLAISHER, Esq., F.R.S., President, in the Chair.

B. Hanumanta Rau, B.A., Acting Head-master, Government Normal School, Madras, was elected a member.

Papers were read by Rev. T. C. Simmons: An Application of Determinants to the Solution of Certain Types of Simultaneous Equations; and by H. M. Jeffery, F.R.S.: On Binodal Quartics (the reading was illustrated by several diagrams).

Mr. Tucker communicated a paper by Professor J. Larmor: On the Flow of Electricity in a System of Linear Conductors.

The following presents were received:—

"Proceedings of the Royal Society," Vol. xxxviii., No. 236.

"Educational Times" for May.

"Proceedings of the Cambridge Philosophical Society," Vol. v., Parts 1, 2, 3, 1885.

"Transactions of the Cambridge Philosophical Society," Vol. xiv., Part 1, 1885.

"Proceedings of the Canadian Institute, Toronto," Third Series, Vol. iii., Fasc. 1, March, 1885.

"Sitzungsberichte der k. Preussischen Akademie der Wissenschaften zu Berlin," xl. to liv., 23 October 1884 to 11 December.

"Journal de l'Ecole Polytechnique," 54th Cahier; Paris 1884.

"Bulletin des Sciences Mathématiques," T. ix., April and May, 1885.

"Beiblätter zu den Annalen der Physik und Chemie," B. ix., St. 4; Leipzig, 1885.

"Atti della R. Accademia dei Lincei," Rendiconti., Vol. i., F. 8, 9, 10; Roma, 1885.

"Über die Theorie der aufeinander abwickelbaren Oberflächen," von J. Weingarten, 4to; Berlin, 1884. (Separatabdruck aus der Festschrift der Königl. Technischen Hochschule zu Berlin.)

On the Flow of Electricity in a System of Linear Conductors.

By Professor J. Larmor.

[*Read May 14th*, 1885.]

1. The analytical determination of the currents that are set up by given steady electro-motive forces in a system of linear conducting bodies has been treated by Kirchhoff,* who takes a separate variable to represent the current flowing in each branch of the system.

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* *Pogg. Annal.*, Bd. 72, 1847; *Gesammelte Abhandlungen*, pp. 22—23.

Maxwell has given a discussion* in which the number of variables is reduced by the equation of continuity, which requires that the total current flowing into any junction is equal to the current flowing from that junction; he takes as variables the potentials of the junctions. This method regards the system as a compound conductor, into which currents are introduced from without; and it gives symmetrical expressions for its resistance measured between any two corners.

We may also express the phenomena in terms of a series of currents flowing in closed circuits whose number is sufficient to completely determine the system. This specification by closed currents is more fundamental in character. When the strengths of the currents are variable owing to electro-magnetic action, this method is the only one very conveniently applicable. For we have, to express the electro-kinetic energy, the ordinary quadratic function of the currents in the separate circuits, the coefficients of which are the coefficients of self and mutual induction of those circuits; and these coefficients are just the quantities that can be most easily determined by known methods of calculation or experiment. Nor is the facility of application disturbed when condensers are included in the system.

In the second edition of Maxwell's *Electricity and Magnetism*, there are two insertions from his lecture notes (§ 282*b*, § 755, on the theory of Wheatstone's Bridge with steady currents, and on the same applied to the determination of coefficients of electro-kinetic induction) which point to this mode of treatment; so that it is likely that the reconstruction of these two theories which (Preface, p. xvi.) the author contemplated would have had some reference to it.

It is proposed to develope concisely the principles of this method of investigation.

2. Consider first the case in which the conductors form a simple network. We may take for circuits the separate meshes of the network, and, if we suppose a barrier drawn across each mesh, we obtain a quasi-polyhedron whose faces are formed by the barriers. Let S, E, F denote the number of its summits, edges, and faces, respectively. It is obvious that only $F-1$ of the circuits are independent, for we can represent the current round the remaining face by a system of equal currents round each of the others, just as Ampère replaced a current in a finite circuit by equal currents in a system of infinitesimal circuits forming a network bounded by the original one.

This number $F-1$ of independent variables is necessarily in agreement with the results of Kirchhoff's method. For there are E branch

* *Elec. and Mag.*, I., §§ 280—82. Also Chrystal, *Encyc. Brit.*, Art. *Electricity*.

currents, subject apparently to S conditions of continuity, but really subject to only $S-1$ independent conditions; because the sum of all the currents flowing into all the junctions is zero—without any condition, as each current flowing into one junction must flow from another. There are therefore $E-S+1$ independent variables, which is equal to the preceding estimate $F-1$ by virtue of Euler's relation

$$S+F=E+2.$$

This discussion throws light on the general case in which the conductors do not form a simple network, and a conductor may therefore necessarily belong to more than two independent circuits. To select a proper system of circuits, begin with any one, and suppose a barrier surface drawn across it; select a second, and suppose it also closed by a barrier; and proceed in this way, subject to the condition that the barriers do not disturb the singly continuous character of the space by forming a closed boundary round any portion of it, *i.e.*, by making it periphractic. The process will terminate of itself when there is no conductor left which is not abutted on by a barrier.

But, having thus secured that the space bounded by the barriers is not periphractic, it must also be made certain that it does not possess any character of multiple continuity (cyclosis); *i.e.*, any closed circuit drawn in the space must be capable of being contracted to a point without cutting through any of the barriers. The space bounded by the surface of an anchor ring is thus doubly continuous, since, to secure simple continuity, it is necessary to draw a barrier surface across the opening of the ring if the outside space is considered,—or across the section of the ring if the inside space is in question. The nature and necessity of this proviso will be made clear by considering again the simple network of this section. We may imagine the system of barriers as forming a continuous sheet; and the removal of one of them will make a hole in this sheet, through which a degree of cyclosis is established. It is clear from these considerations that every degree of cyclosis implies the absence of a necessary independent variable, which can be supplied by adding a new barrier closing the corresponding circuit.

With a notation corresponding to that given above for a network, we have in this case for the number of independent circuits the value F, which must on Kirchhoff's principles be equivalent to $E-S+1$; and we thus come upon the theorem that for a polygonal system of barriers which does not impair the singly continuous character of the space, by either periphraxy or cyclosis,

$$S+F=E+1.$$

This is a particular case of Listing's generalization of Euler's theorem, now equally obvious, which asserts that, if the system of barriers divide space into R unconnected regions, each of them singly continuous, then

$$S+F=E+R;$$

for removing each superfluous barrier diminishes the number of regions by one.

The general theorem, as given by Listing,* takes account of the corrections to be applied to this formula when the periphractic and cyclomatic numbers of the system are given constants, different from zero; but with these we are not at present concerned, though the ideas here employed would probably yield a simple method for their discussion.

The particular case of the theorem at which we have arrived verifies the correctness of the method that has been given for choosing the independent circuits of the current system; and this plan of construction by barriers has the advantage of easily showing the precise amount of liberty there is in the selection of the circuits.

3. Having thus selected the circuits, let them be denoted by the natural numbers 1, 2, 3 ... n; the currents circulating in them by $C_1, C_2 \ldots C_n$, their resistances by $R_1, R_2 \ldots R_n$, the electro-motive forces *placed* in them by $E_1, E_2 \ldots E_n$ measured positive in the directions of the currents; and let also the current in the conductor pq which is common to the circuits p and q be denoted by C_{pq} and the resistance of that conductor by R_{pq} and the electro-motive force *placed* in it by E_{pq}: so that we have

$$R_p = \quad R_{p1}+R_{p2}+\ldots+R_{pn} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$E_p = \pm E_{p1}\pm E_{p2}\ldots\ldots\pm E_{pn} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The signs in this last relation are determined in each special case by the directions of the component electro-motive forces as compared with E_p. In the case of a simple network we can secure that the positive directions of circulation of all the currents as seen from one side of the network shall be the same, and we shall then have also relations of the form

$$C_{pq} = C_q - C_p \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

But, when the system does not form a simple network, the same conductor may be common to three or more circuits, say pqr ...; and then

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* J. B. Listing, "Der Census raümlicher Complexe," *Göttingen Abhandlungen*, Band x., 1861-2.

the current in that conductor is denoted by C_{pq} or C_{pr} or C_{qr}, where, the signs being properly determined,

$$C_{pq} = \pm C_p \pm C_q \pm C_r \quad \text{.............................(4).}$$

It will not be necessary to have an explicit notation for the case in which more than one conductor is common to two circuits.

With this notation, the expression for the heat generated per second by steady currents in the conductors can be obtained as follows:

$$H = \Sigma R_{pq} C_{pq}^2,$$

the summation extending over all the conductors of the system,

$$\begin{aligned} &= \Sigma R_{pq} (C_p - C_q)^2 \\ &= R_1 C_1^2 + R_2 C_2^2 + \dots \\ &\qquad \pm 2R_{pq} C_p C_q \pm \dots \quad \text{...............(5),} \end{aligned}$$

the latter signs being all negative for a simple network; while for any other case the rule is that, when C_p and C_q are taken to flow in the same direction along the conductor pq, the sign of the corresponding term is positive. Terms involving products of all pairs of contiguous closed currents are included in the expression.

4. The theorem connecting the electro-motive forces with H may now be investigated by the method common to all analyses which turn upon quadratic functions of this kind. Let accented letters denote any other system of electro-motive forces and corresponding currents imposed upon the same system of conductors: then we have

$$\begin{aligned} &\Sigma C_p E'_p \\ &= \Sigma C_{pq} e'_{pq}, \end{aligned}$$

where e'_{pq} represents the total *gradual* fall of potential along the conductor pq in the direction of the current C_{pq},

$$\begin{aligned} &= \Sigma \frac{e_{pq} e'_{pq}}{R_{pq}}, \text{ by Ohm's law,} \\ &= \Sigma C'_p E_p, \text{ by symmetry.} \end{aligned}$$

Therefore, if $C'_p = C_p + \delta C_p$, $E'_p = E_p + \delta E_p$, we have

$$\Sigma C_p \delta E_p = \Sigma E_p \delta C_p \quad \text{.............................(6).}$$

Now $$H = \Sigma C_p E_p,$$

therefore $$\delta H = \Sigma C_p \delta E_p + \Sigma E_p \delta C_p.$$

Hence finally, by (6),

$$\delta H = 2\Sigma E_p \delta C_p,$$

or, proceeding from finite increments to infinitesimals,

$$\frac{dH_c}{dC_p} = 2E_p, \quad \frac{dH_c}{dC_q} = 2E_q, \text{ \&c.} \qquad (7).$$

where H_c is H expressed as in (5) by a function of C_p, C_q, ...

Again, when, by means of (7), C_p, C_q ... are eliminated from the expression for H_c in (5), we have for H a quadratic function H_e of the electro-motive forces, and then

$$\frac{dH_e}{dE_p} = 2C_p, \quad \frac{dH_e}{dE_q} = 2C_q, \text{ \&c.} \qquad (8).$$

5. The equations (7) are the linear system to which we are led for the determination of the currents. Writing them in full, we have

$$\left.\begin{array}{c} R_1C_1 - R_{12}C_2 - R_{13}C_3 \ldots - R_{1n}C_n = E_1 \\ -R_{21}C_1 + R_2C_2 - R_{23}C_3 \ldots - R_{2n}C_n = E_2 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{array}\right\} \qquad (9),$$

a symmetrical system, in which the proper signs are here supposed to be given to the R's with double suffixes, viz., all positive in the case of a simple network, and in other cases according to the rule already given. The solution can be expressed by symmetrical determinants in the ordinary manner, and by means of (4) we can obtain at once an expression for the total current in any separate conductor.

We may thus also obtain the condition that an electro-motive force in one conductor may not give rise to a current in a certain other. But for such purposes it is more convenient to employ the reciprocal expression for H in terms of the E's. To obtain this, proceed in the usual way, by joining on to (9) the equation

$$E_1C_1 + E_2C_2 + \ldots + E_nC_n = H \qquad (10),$$

and by linear elimination we find

$$H = -\frac{1}{\Delta}\left|\begin{array}{ccccc} R_1 & -R_{12} & -R_{13} \ldots & -R_{1n} & E_1 \\ -R_{21} & R_2 & -R_{23} \ldots & -R_{2n} & E_2 \\ -R_{31} & \ldots & \ldots & \ldots & \\ \ldots & \ldots & \ldots & \ldots & \\ \ldots & \ldots & \ldots & \ldots & \\ E_1 & E_2 & E_3 \ldots & E_n & 0 \end{array}\right| \qquad (11),$$

in which Δ is the discriminant of the expression for H_c, *i.e.*, is the minor of the last (zero) constituent of this determinant.

We may write this quadratic expression in the form

$$H_e = K_1 E_1^2 + K_2 E_2^2 + \dots + K_n E_n^2$$
$$+ 2K_{12} E_1 E_2 + 2K_{23} E_2 E_3 + \dots \quad \dots\dots\dots\dots\dots (12),$$

in which K_1, K_{12}, ... are immediately written down as minors of the determinant Δ. We have then, by (8),

$$\left.\begin{array}{l} K_1 E_1 + K_{12} E_2 + K_{13} E_3 + \dots + K_{1n} E_n = C_1 \\ K_{21} E_1 + K_2 E_2 + K_{23} E_3 + \dots + K_{2n} E_n = C_2 \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \end{array}\right\} \dots\dots\dots\dots (13).$$

The constants K_1, K_{12} ... are clearly coefficients of conductivity.

Now suppose the only electro-motive force considered is E_{pq} in the conductor pq. If this conductor abuts on two circuits only, we have $E_p = \pm E_q = E_{pq}$ and the other E's zero; if on three or more, we have $E_p = \pm E_q = \pm E_r = \dots = E_{pq}$, in which the signs have been predetermined; while, if it abuts on only one circuit, we have only one fundamental E. To determine the current in the conductor lm, we have

$$C_{lm} = C_l \pm C_m \pm C_n \pm \dots \quad \dots\dots\dots\dots\dots\dots\dots (14).$$

We find thus, by (8) or (13),

$$C_{lm} = E_{pq} \left\{\begin{array}{l} \quad K_{lp} \pm K_{lq} \pm K_{lr} \dots \\ \pm (K_{mp} \pm K_{mq} \pm K_{mr} \dots) \\ \pm (K_{np} \pm K_{nq} \pm K_{nr} \dots) \end{array}\right\} \dots\dots\dots\dots\dots (15),$$

which reduces, for the case of a simple network, to

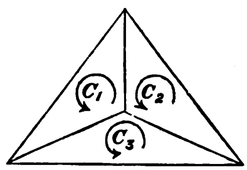
$$C_{lm} = E_{pq} (K_{lp} - K_{lq} - K_{mp} + K_{mq}) \dots\dots\dots\dots\dots\dots (16).$$

We conclude that an electro-motive force in pq will produce no current in lm if the coefficient of E_{pq} in this result is zero; and, as the coefficient remains unaltered when lm and pq change places, we see that equal currents are produced in either conductor by unit electro-motive force in the other.

The two conductors are therefore conjugate to one another if this coefficient is zero.

These results are of the same general form as the ones obtained in Maxwell's investigation; and the similarity is intelligible when it is remembered that the two investigations are in a sense reciprocal to

each other, the variables in the one case being related to the corners and in the other case to the faces of the diagram.



6. A simple case is that of the six conductors of Wheatstone's Bridge,* which form a network, as in the diagram. We have, by § 3,

$$H_c = R_1 C_1^2 + R_2 C_2^2 + R_3 C_3^2 - 2R_{12} C_1 C_2 - 2R_{23} C_2 C_3 - 2R_{31} C_3 C_1 \quad \ldots\ldots(17).$$

The reciprocal function is

$$H_e = -\frac{1}{\Delta}\begin{vmatrix} R_1 & -R_{12} & -R_{13} & E_1 \\ -R_{21} & R_2 & -R_{23} & E_2 \\ -R_{31} & -R_{32} & R_3 & E_3 \\ E_1 & E_2 & E_3 & 0 \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots(18),$$

$$= \frac{1}{\Delta}\left\{(R_2 R_3 - R_{23}^2) E_1^2 + \ldots + \ldots - 2(R_{12} R_{13} - R_1 R_{23}) E_2 E_3 + \ldots + \ldots\right\} \quad \ldots\ldots\ldots\ldots\ldots(19),$$

Δ being the discriminant of H_c.

The coefficients in this expression for H_e are the conductivities of the system. The coefficient of E_1^2 represents the current in E_1 that would be produced by unit electro-motive force in E_1 *alone*; the coefficient of $E_2 E_3$ represents double the current that would be produced in either circuit E_2 or E_3 by unit electro-motive force in the other one *alone*.

7. The form of this analytical theory is exactly analogous to that of the theory of initial motions in a system of bodies. We have therefore an analogue of Sir W. Thomson's theorem of least energy with certain given imposed velocities; and also one of Bertrand's theorem of greatest energy with certain given imposed impulses, subject to the condition that the only allowable variations of the motion are those caused by pure constraint, or by such other impulses as do no work in the process: viz., we have the theorems—

1°. When given currents are introduced into a system, they distribute themselves in such a manner that the heat developed in the conductors is the least possible.

2°. When given electro-motive forces are introduced into a system of conductors, the heat actually generated by the currents is greater

* Maxwell's *Elec. and Mag.*, Vol. I., 2nd ed., §§ 282*b*, 347; *Elementary Electricity*, p. 206.

than it would be if any of the conductors were removed* ; for the removal of a conductor is equivalent to the introduction of such an electro-motive force in it as reduces its current to zero.

8. The general theory of linear conductors, and the conjugate relations involved, clearly apply also to cases in which the conducting system is partly linear and partly continuous in other dimensions. If the system include any continuous conductor (with or without helical property) with a number of electrodes on its surface, we can replace that part of it by a system of linear conductors of the proper resistances connecting all the electrodes in pairs,—their resistances being determinable by calculation or experiment from the shape of the conductor. For, from the linearity of the law of conduction, it is permissible to suppose different current systems.

Again, we can imagine a continuous conductor such that in its ultimate structure it is composed of a thicket of interlacing conducting filaments whose cross-sections are negligible in comparison with their lengths; though this will not, of course, be a probable representation of the constitution of an ordinary conducting solid. We can show that, for a small right-angled element of such a body, the equations of conduction will be self-conjugate; *i.e.*, there will be no helical coefficient. For suppose (for purposes of analysis) such an element cut out of the solid, and its faces backed up by six perfectly conducting plates, and the opposite pairs of faces connected by conductors of no resistance in which electro-motive faces E_1, E_2, E_3 are placed. We shall then have a linear system, in the sense of the above theory; and H_e will be a quadratic function of these three electro-motive forces; and the corresponding currents across the faces of the element will be derived from H_e by differentiation, according to (8), with respect to E_1, E_2, E_3, which proves the proposition.

9. It remains to indicate briefly how the method here used may be applied to the general case of currents variable owing to electro-dynamic action or to the gradual charge of condensers whose terminals are connected with the system at given points.

We may treat such a condenser as a branch of the system whose resistance is infinite; or, if we make allowance for the small degree of conductivity which may exist between its faces, it will be a conductor of resistance very large.

As the currents are now variable we shall want to change their notation. Let x_1, x_2, ... x_n represent the *integral* currents that have

* Lord Rayleigh, *Phil. Mag.*, Vol. XLVIII., 1875.

flowed round the specifying circuits from the beginning of the motion; then, with the usual fluxional notation, $\dot{x}_1, \dot{x}_2, \dots \dot{x}_n$ will represent the currents in those circuits at the instant considered.

Following the notation of § 3, let rs denote a condenser branch; if k_{rs} denote the capacity of the condenser, the energy stored *from the system* in its charge and in the charges of the others, if any, will be

$$V = \Sigma \tfrac{1}{2} k_{rs}^{-1} x_{rs}^2 = \Sigma \tfrac{1}{2} k_{rs}^{-1} (x_r \pm x_s \pm \dots)^2 \quad \dots\dots\dots\dots\dots (20)$$

by (4).

The electro-kinetic energy due to the motion of the conductors will be

$$T = \tfrac{1}{2} M_1 \dot{x}_1^2 + \tfrac{1}{2} M_2 \dot{x}_2^2 + \dots + M_{12} \dot{x}_1 \dot{x}_2 + \dots\dots\dots\dots\dots (21),$$

where M_1, M_2 ... are the coefficients of self-induction of the respective circuits, and M_{12}, ... are the coefficients of mutual induction of the different pairs of circuits.

If there is any part of the electro-kinetic energy due to the influence of external fixed systems, it will be represented by a function of $\dot{x}_1, \dot{x}_2$ ... of the first degree, which must be added on to T.

[In the general electro-kinetic theory mentioned below, the coefficients of this linear function will involve constant electro-kinetic momenta whose corresponding variables have been eliminated from the expression for the energy; and we shall thus have an example of the general dynamical equations with ignored co-ordinates.*]

The expression for the amount of energy that runs down into heat per second, owing to the resistances of the wires, is, as in (5),

$$H = R_1 \dot{x}_1^2 + R_2 \dot{x}_2^2 + \dots \pm 2R_{12} \dot{x}_1 \dot{x}_2 + \dots \quad \dots\dots\dots\dots\dots (22).$$

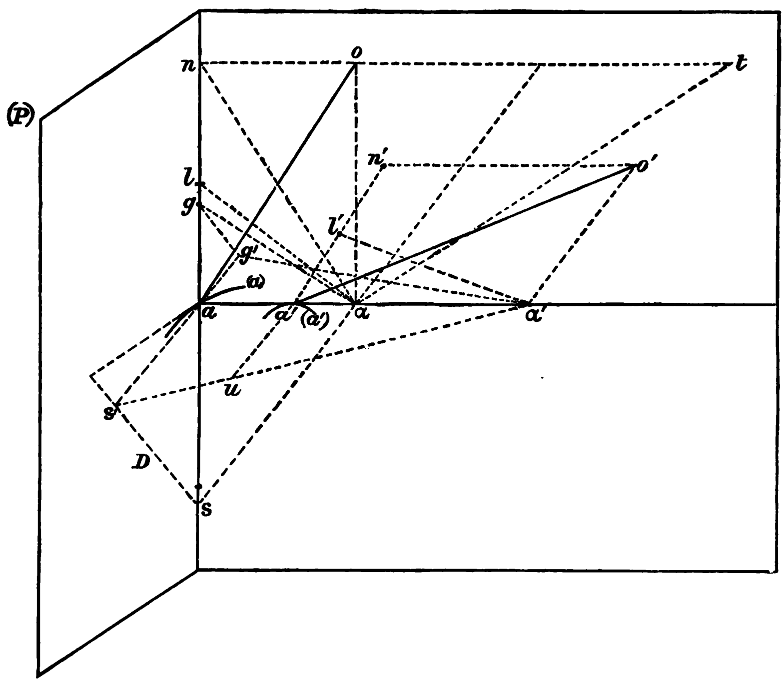
And we have also the ordinary quadratic function for the kinetic energy $\mathfrak{T}$ of the masses of the system in terms of the generalized velocities.

From these four expressions the equations of the currents $\dot{x}_1, \dot{x}_2$ ... may be found by the principle of energy (as used originally by Helmholtz and Sir W. Thomson) combined with Ohm's law of currents; and it is well known, as Maxwell has shown, that these equations are the same as those of a purely dynamical system, of which x_1, x_2 ... are additional coordinates, $\dot{x}_1, \dot{x}_2$ ... the corresponding velocities, T the corresponding part of the kinetic energy, $-V$ the corresponding part of the potential energy, and H the corresponding *dissipation function* of Lord Rayleigh.†

* Thomson and Tait's *Natural Philosophy*, Vol. I., Part I., p. 320.

† *Proc. Lond. Math. Soc.*, May, 1873; *Theory of Sound*, I., § 81.

La tangente en α à (α) est la perpendiculaire αt à αn.



Appelons (S_N) la surface formée par les normales principales communes aux courbes (a) et (a'). Le plan normal en a à (a) est normal en ce point à (S_N) et touche cette surface au point α. Le plan normal à (a), au point de cette courbe infiniment voisin de a, touche (S_N) au point de (α) qui est infiniment voisin de α. Ces deux plans normaux infiniment voisins se coupent suivant αo et la droite qui joint les points où ils touchent (S_N) est αt; donc:

Les droites αt, αo sont deux tangentes conjuguées de (S_N).

On connait pour le point α de (S_N) les deux tangentes conjuguées αt, αo et la droite αa qui est une des asymptotes de l'indicatrice de (S_N) en α; il est alors facile de construire l'autre asymptote de cette indicatrice. Il suffit pour cela de joindre le point α au point milieu du segment ot, qui est parallèle à αa; ou encore, d'élever du point α une perpendiculaire à la droite αl qui passe par le point l, milieu de an.

On voit ainsi comment la connaissance du centre o de la sphère

osculatrice en a à (a) entraîne la connaissance de l'asymptote de l'indicatrice de (S_N) au point α, centre de courbure de (a).

Inversement, la connaissance de cette asymptote permet de construire le centre o. Nous n'avons d'après cela qu'à déterminer l'asymptote de l'indicatrice de (S_N) en α' pour obtenir le centre o' de la sphère osculatrice de (a') en a'.

Les asymptotes des indicatrices de (S_N) en a et a' sont les tangentes en ces points à (a) et (a'). Nous connaissons alors les asymptotes des indicatrices de (S_N) aux trois points a, a', α et nous pouvons construire l'asymptote de l'indicatrice en un point quelconque de aa'. Toutes ces droites appartiennent, en effet, à un même hyperboloïde qui est l'hyperboloïde osculateur de (S_N) le long de aa'.

Effectuons les constructions en faisant usage du plan (P) mené du point a perpendiculairement à aa'. Ce plan contient déjà l'asymptote de l'indicatrice en a, il ne coupe plus alors l'hyperboloïde osculateur que suivant une droite D. Cette droite D est le lieu des traces sur (P) des asymptotes des indicatrices de (S_N) pour les points de aa'. Soit s la trace sur (P) de l'asymptote de l'indicatrice en α. Menons de s une parallèle à l'asymptote relative au point a', c'est à dire à la tangente en a' à (a'). Nous obtenons ainsi la droite D. L'asymptote relative au point α' est dans le plan tangent en ce point à (S_N), c'est à dire dans le plan normal en a' à (a'). La trace de ce plan sur (P) est alors la perpendiculaire as' abaissée du point a sur D, et *la droite $\alpha's'$ est l'asymptote de l'indicatrice de (S_N) en α'.*

Connaissant cette droite, il est facile de déterminer o'. Pour cela on mène $a'u$ parallèlement à as', et, du point α', dans le plan $a\,\alpha's'$, on élève la perpendiculaire $\alpha'l'$ à $\alpha's'$: ces deux droites se coupent en l'. On prolonge le segment $a'l'$ de sa propre longueur jusqu'en n' et l'on construit le rectangle $n'a'\alpha'o'$: *le sommet o' de ce rectangle est le centre de la sphère osculatrice demandé.*

Nous avons ainsi une construction du centre o'; nous allons la simplifier après avoir fait remarquer, comme conséquence du tracé de αs, que la perpendiculaire abaissée du centre de courbure α sur le rayon ao de la sphère osculatrice rencontre sa en un point g qui est tel que ag est la moitié de as. On obtient de même pour la courbe a' un point g' tel que ag' et la moitié de as'. La droite gg' est alors parallèle à D, c'est à dire à la tangente en a' à (a'). On a donc cette propriété :

Les perpendiculaires, abaissées respectivement des centres de courbure α, α' sur les rayons ao, $a'o'$, des sphères osculatrices à (a) et (a'), rencontrent le plan, mené en a perpendiculairement à aa', en deux points g, g' : le tri-

angle agg′ est rectangle en g′ et la droite gg′ est parallèle à la tangente en a′ à (a′).

Cette propriété, qu'on peut énoncer en employant un plan perpendiculaire en a' à la normale commune, établit une liaison géométrique simple entre les centres o et o' qui permet de construire l'un de ces points lors qu'on connait l'autre.

New Relations between Bipartite Functions and Determinants, with a Proof of Cayley's Theorem in Matrices. By THOMAS MUIR, LL.D.

[*Read April 2nd*, 1885.]

1. A bipartite function may be denoted as in the examples,

$$\begin{array}{cc|cc|c} a_1 & a_2 & & & \\ \hline b_1 & c_1 & d_1 & d_2 & \\ b_2 & c_2 & e_1 & e_2 & \\ \cline{3-4} & & f_1 \; g_1 & & h_1 \\ & & f_2 \; g_2 & & k_1 \end{array} \qquad \text{or} \qquad \begin{array}{cc|cc|c} & & f_2 & g_2 & k_1 \\ a_1 & a_2 & f_1 & g_1 & h_1 \\ \hline b_1 & c_1 & d_1 & d_2 & \\ b_2 & c_2 & e_1 & e_2 & \end{array},$$

and

$$\begin{array}{ccc|ccc} a_1 & a_2 & a_3 & h_1 & k_1 & l_1 \\ \hline b_1 & c_1 & d_1 & e_1 & e_2 & e_3 \\ b_2 & c_2 & d_2 & f_1 & f_2 & f_3 \\ b_3 & c_3 & d_3 & g_1 & g_2 & g_3 \end{array},$$

the elements of the function being distributed in arrays, viz., (1) an initial one-line array of n elements, (2) any number of square arrays each of $n \, . \, n$ elements, (3) a final one-line array of n elements; and the arrays being separated by bars, the first of which is parallel to the initial one-line array, and each of the others at right angles to the one preceding it.

The ordinary algebraical expression of the function consists of all the terms that can be formed by taking the product of as many elements as there are arrays, one from each, subject to the condition that the element to be taken from any one array must be in the same row or column with the element taken from the preceding array, and

in the same column or row with the element taken from the following array. Thus,

$$\frac{a_1,\ a_2,\ a_3,\ a_4}{x_1,\ y_1,\ z_1,\ w_1} \equiv a_1x_1+a_2y_1+a_3z_1+a_4w_1,$$

$$\equiv (a_1\,a_2\,a_3\,a_4 \between x_1\,y_1\,z_1\,w_1)\ ;$$

and

$$\begin{array}{cc|cc} \multicolumn{2}{c}{a_1\ \ a_2} & & \\ \hline b_1 & c_1 & d_1 & d_2 \\ b_2 & c_2 & e_1 & e_2 \\ \cline{3-4} & & \multicolumn{2}{c}{f_1\ \ f_2} \end{array} \equiv \begin{array}{l} a_1b_1d_1f_1+a_1b_1d_2f_2+a_1b_2e_1f_1+a_1b_2e_2f_2 \\ +a_2c_1d_1f_1+a_2c_1d_2f_2+a_2c_2e_1f_1+a_2c_2e_2f_2. \end{array}$$

2. A bipartite is expressible as a sum of products of two factors, each first factor being an element taken from the initial line, and its co-factor the bipartite of lower degree got from the original bipartite by deleting the initial line and those elements of the adjacent square which are not collinear with the said element. Thus,

$$\begin{array}{ccc|ccc} a_1 & a_2 & a_3 & h_1 & k_1 & l_1 \\ \hline b_1 & c_1 & d_1 & e_1 & e_2 & e_3 \\ b_2 & c_2 & d_2 & f_1 & f_2 & f_3 \\ b_3 & c_3 & d_3 & g_1 & g_2 & g_3 \end{array}$$

$$= a_1 \cdot \begin{array}{ccc|c} h_1 & k_1 & l_1 & \\ \hline e_1 & e_2 & e_3 & b_1 \\ f_1 & f_2 & f_3 & b_2 \\ g_1 & g_2 & g_3 & b_3 \end{array} + a_2 \cdot \begin{array}{ccc|c} h_1 & k_1 & l_1 & \\ \hline e_1 & e_2 & e_3 & c_1 \\ f_1 & f_2 & f_3 & c_2 \\ g_1 & g_2 & g_3 & c_3 \end{array} + a_3 \cdot \begin{array}{ccc|c} h_1 & k_1 & l_1 & \\ \hline e_1 & e_2 & e_3 & d_1 \\ f_1 & f_2 & f_3 & d_2 \\ g_1 & g_2 & g_3 & d_3 \end{array}.$$

This recurrent law of formation is all that need here be given in regard to bipartites, the properties of which have been very fully dealt with in a memoir about to appear in the *Transactions of the Royal Society of Edinburgh.*

3. One connecting link between the theory of bipartites and the theory of determinants has been pointed out in the said memoir; viz., that the elements of the determinant which is the product of n determinants are bipartites of the n^{th} degree. Thus,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \cdot \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} = \begin{vmatrix} \dfrac{a_1\,a_2\,a_3}{\alpha_1\,\beta_1\,\gamma_1} & \dfrac{a_1\,a_2\,a_3}{\alpha_2\,\beta_2\,\gamma_2} & \dfrac{a_1\,a_2\,a_3}{\alpha_3\,\beta_3\,\gamma_3} \\ \dfrac{b_1\,b_2\,b_3}{\alpha_1\,\beta_1\,\gamma_1} & \dfrac{b_1\,b_2\,b_3}{\alpha_2\,\beta_2\,\gamma_2} & \dfrac{b_1\,b_2\,b_3}{\alpha_3\,\beta_3\,\gamma_3} \\ \dfrac{c_1\,c_2\,c_3}{\alpha_1\,\beta_1\,\gamma_1} & \dfrac{c_1\,c_2\,c_3}{\alpha_2\,\beta_2\,\gamma_2} & \dfrac{c_1\,c_2\,c_3}{\alpha_3\,\beta_3\,\gamma_3} \end{vmatrix},$$

and $\qquad | a_1 b_2 c_3 | \cdot | \alpha_1 \beta_2 \gamma_3 | \cdot | x_1 y_2 z_3 |$

$$= \left| \begin{array}{ccc} a_1 \; a_2 \; a_3 & b_1 \; b_2 \; b_3 & c_1 \; c_2 \; c_3 \\ \hline \begin{array}{ccc|c} \alpha_1 & \beta_1 & \gamma_1 & x_1 \\ \alpha_2 & \beta_2 & \gamma_2 & y_1 \\ \alpha_3 & \beta_3 & \gamma_3 & z_1 \end{array} & \begin{array}{ccc|c} \alpha_1 & \beta_1 & \gamma_1 & x_2 \\ \alpha_2 & \beta_2 & \gamma_2 & y_2 \\ \alpha_3 & \beta_3 & \gamma_3 & z_2 \end{array} & \begin{array}{ccc|c} \alpha_1 & \beta_1 & \gamma_1 & x_3 \\ \alpha_2 & \beta_2 & \gamma_2 & y_3 \\ \alpha_3 & \beta_3 & \gamma_3 & z_3 \end{array} \end{array} \right| .$$

Our present object is to establish other relations connecting the two kinds of functions, and to make application of one of these relations to Cayley's identity in the theory of matrices.

4. Starting from the manifest identity

$$\frac{a_1, a_2, a_3, \ldots a_n}{a_r, b_r, c_r, \ldots l_r} - (a_1 + b_2 + c_3 + \ldots + l_n)\, a_r$$
$$= | a_1 a_r | + | a_2 b_r | + | a_3 c_r | + \ldots + | a_n l_r | ,$$

and restricting ourselves, for convenience in writing, to the case where $n = 4$, we have

$$\frac{a_1, a_2, a_3, a_4}{a_1, b_1, c_1, d_1} - (a_1 + b_2 + c_3 + d_4)\, a_1 = - | a_1 b_2 | - | a_1 c_3 | - | a_1 d_4 | \; \ldots (\alpha),$$

$$\frac{a_1, a_2, a_3, a_4}{a_2, b_2, c_2, d_2} - (a_1 + b_2 + c_3 + d_4)\, a_2 = - | a_2 c_3 | - | a_2 d_4 | \ldots\ldots\ldots\ldots\ldots (\beta),$$

$$\frac{a_1, a_2, a_3, a_4}{a_3, b_3, c_3, d_3} - (a_1 + b_2 + c_3 + d_4)\, a_3 = \quad | a_2 b_3 | - | a_3 d_4 | \ldots\ldots\ldots\ldots\ldots (\gamma),$$

$$\frac{a_1, a_2, a_3, a_4}{a_4, b_4, c_4, d_4} - (a_1 + b_2 + c_3 + d_4)\, a_4 = \quad | a_2 b_4 | + | a_3 c_4 | \; \ldots\ldots\ldots\ldots\ldots (\delta).$$

If now we multiply both sides of these by a_1, b_1, c_1, d_1 respectively, add, and, for shortness, write Σa_1 for $a_1 + b_2 + c_3 + d_4$, there results

$$\begin{array}{cccc|c} a_1 & a_2 & a_3 & a_4 & \\ \hline a_1 & b_1 & c_1 & d_1 & a_1 \\ a_2 & b_2 & c_2 & d_2 & b_1 \\ a_3 & b_3 & c_3 & d_3 & c_1 \\ a_4 & b_4 & c_4 & d_4 & d_1 \end{array} - \frac{a_1, a_2, a_3, a_4}{a_1, b_1, c_1, d_1} \sum a_1 = - \{ | a_1 b_2 | + | a_1 c_3 | + | a_1 d_4 | \} \, a_1$$
$$+ \begin{vmatrix} 0 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} 0 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ d_1 & d_2 & d_4 \end{vmatrix} + \begin{vmatrix} 0 & a_3 & a_4 \\ c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{vmatrix} ;$$

and this, by the addition of

$$\{ | a_1 b_2 | + | a_1 c_3 | + | a_1 d_4 | + | b_2 c_3 | + | b_2 d_4 | + | c_3 d_4 | \} \, a_1$$

or

$$a_1 \sum | a_1 b_2 |$$

to both sides, becomes

$$\begin{array}{cccc|c} a_1 & a_2 & a_3 & a_4 & \\ \hline a_1 & b_1 & c_1 & d_1 & a_1 \\ a_2 & b_2 & c_2 & d_2 & b_1 \\ a_3 & b_3 & c_3 & d_3 & c_1 \\ a_4 & b_4 & c_4 & d_4 & d_1 \end{array} - \frac{a_1, a_2, a_3, a_4}{a_1, b_1, c_1, d_1}\sum a_1 + a_1 \sum |\, a_1 b_2 \,|$$

$$= |\, a_1 b_2 c_3 \,| + |\, a_1 b_2 d_4 \,| + |\, a_1 c_3 d_4 \,| \;\ldots\ldots (\alpha').$$

By using, instead of a_1, b_1, c_1, d_1, the multipliers a_2, b_2, c_2, d_2, and then proceeding as before, there results

$$\begin{array}{cccc|c} a_1 & a_2 & a_3 & a_4 & \\ \hline a_1 & b_1 & c_1 & d_1 & a_2 \\ a_2 & b_2 & c_2 & d_2 & b_2 \\ a_3 & b_3 & c_3 & d_3 & c_2 \\ a_4 & b_4 & c_4 & d_4 & d_2 \end{array} - \frac{a_1, a_2, a_3, a_4}{a_2, b_2, c_2, d_2}\sum a_1 + a_2 \sum |\, a_1 b_2 \,| = |\, a_2 c_3 d_4 \,| \;\ldots\ldots\ldots (\beta').$$

Similarly, we find

$$\begin{array}{cccc|c} a_1 & a_2 & a_3 & a_4 & \\ \hline a_1 & b_1 & c_1 & d_1 & a_3 \\ a_2 & b_2 & c_2 & d_2 & b_3 \\ a_3 & b_3 & c_3 & d_3 & c_3 \\ a_4 & b_4 & c_4 & d_4 & d_3 \end{array} - \frac{a_1, a_2, a_3, a_4}{a_3, b_3, c_3, d_3}\sum a_1 + a_3 \sum\; a_1 b_2 \,| = -|\, a_2 b_3 d_4 \,| \;\ldots\ldots (\gamma),$$

and

$$\begin{array}{cccc|c} a_1 & a_2 & a_3 & a_4 & \\ \hline a_1 & b_1 & c_2 & d_1 & a_4 \\ a_2 & b_2 & c_2 & d_2 & b_4 \\ a_3 & b_3 & c_3 & d_3 & c_4 \\ a_4 & b_4 & c_4 & d_4 & d_4 \end{array} - \frac{a_1, a_2, a_3, a_4}{a_4, b_4, c_4, d_4}\sum a_1 + a_4 \sum |\, a_1 b_2 \,| = |\, a_2 b_3 c_4 \,| \;\ldots\ldots\ldots (\delta').$$

Treating, in their turn, these four equations (α'), (β'), (γ'), (δ') as (α), (β), (γ), (δ) were treated, the multipliers being a_1, b_1, c_1, d_1, we obtain, on the left-hand side,

$$\begin{array}{cccc|cccc} a_1 & a_2 & a_3 & a_4 & a_1 & b_1 & c_1 & d_1 \\ \hline a_1 & b_1 & c_1 & d_1 & a_1 & a_2 & a_3 & a_4 \\ a_2 & b_2 & c_2 & d_2 & b_1 & b_2 & b_3 & b_4 \\ a_3 & b_3 & c_3 & d_3 & c_1 & c_2 & c_3 & c_4 \\ a_4 & b_4 & c_4 & d_4 & d_1 & d_2 & d_3 & d_4 \end{array} - \begin{array}{cccc|c} a_1 & a_2 & a_3 & a_4 & \\ \hline a_1 & b_1 & c_1 & d_1 & a_1 \\ a_2 & b_2 & c_2 & d_2 & b_1 \\ a_3 & b_3 & c_3 & d_3 & c_1 \\ a_4 & b_4 & c_4 & d_4 & d_1 \end{array} \sum a_1 + \frac{a_1, a_2, a_3, a_4}{a_1, b_1, c_1, d_1}\sum |\, a_1 b_2 \,|$$

and on the right

$$a_1\left\{|\,a_1 b_2 c_3\,| + |\,a_1 b_2 d_4\,| + |\,a_1 c_3 d_4\,|\right\} - \left|\begin{array}{cccc} 0 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_3 \\ d_1 & d_2 & d_3 & d_4 \end{array}\right|$$

or $\quad a_1\left\{|\,a_1 b_2 c_3\,| + |\,a_1 b_2 d_4\,| + |\,a_1 c_3 d_4\,| + |\,b_2 c_3 d_4\,|\right\} - |\,a_1 b_2 c_3 d_4\,|$;

so that we have the identity

$$\frac{a_1\ a_2\ a_3\ a_4}{\begin{array}{cccc} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{array}} \left| \frac{a_1\ b_1\ c_1\ d_1}{\begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{array}} \right. - \frac{a_1\ a_2\ a_3\ a_4}{\begin{array}{cccc|c} a_1 & b_1 & c_1 & d_1 & a_1 \\ a_2 & b_2 & c_2 & d_2 & b_1 \\ a_3 & b_3 & c_3 & d_3 & c_1 \\ a_4 & b_4 & c_4 & d_4 & d_1 \end{array}} \sum a_1$$

$$+ \frac{a_1, a_2, a_3, a_4}{a_1, b_1, c_1, d_1} \sum |\,a_1 b_2\,| - a_1 \sum |\,a_1 b_2 c_3\,| = -|\,a_1 b_2 c_3 d_4\,| \quad \ldots\ (\alpha'').$$

The three other identities related to this as (β'), (γ'), (δ') are related to (α') have their right-hand member 0. The four may be stated as one, and it is useful to do so in view of what is to follow. Taking, then, a_r, b_r, c_r, d_r as multipliers, we obtain, from (α'), (β'), (γ'), (δ'), the identity

$$\frac{a_1\ a_2\ a_3\ a_4}{\begin{array}{cccc} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{array}} \left| \frac{a_r\ b_r\ c_r\ d_r}{\begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{array}} \right. - \frac{a_1\ a_2\ a_3\ a_4}{\begin{array}{cccc|c} a_1 & b_1 & c_1 & d_1 & a_r \\ a_2 & b_2 & c_2 & d_2 & b_r \\ a_3 & b_3 & c_3 & d_3 & c_r \\ a_4 & b_4 & c_4 & d_4 & d_r \end{array}} \sum a_1$$

$$+ \frac{a_1, a_2, a_3, a_4}{a_r, b_r, c_r, d_r} \sum |\,a_1 b_2\,| - a_r \sum |\,a_1 b_2 c_3\,| = - \left|\begin{array}{cccc} a_r & a_2 & a_3 & a_4 \\ b_r & b_2 & b_3 & b_4 \\ c_r & c_2 & c_3 & c_4 \\ d_r & d_2 & d_3 & d_4 \end{array}\right|.$$

5. It is instructive to compare the series of identities (α), (α'), (α''), and note the law of growth, so to speak. In the mature form (α'') the bipartite factors of the terms form a regular descending series; that in the first term being of the degree 4, and that in the last term of the degree 0; while the determinantal factors ascend from the degree 0 to the degree 4. In every bipartite factor the initial line is a_1, a_2, a_3, a_4, and the final line a_1, b_1, c_1, d_1. Further, all the square

arrays are alike; so that, if we use a single symbol, such as Δ, to stand for a square array, and annex to it a number to indicate how many square arrays there are, we obtain a very short and suggestive notation for the particular bipartites here occurring. Thus, the theorem (a'') for the case of the sixth degree might stand as follows:—

$$
\begin{aligned}
&\frac{a_1, a_2, \ldots, a_6 \mid a_1, b_1, \ldots, f_1}{\Delta^4} \\
-&\frac{a_1, a_2, \ldots, a_6 \mid a_1, b_1, \ldots, f_1}{\Delta^3} \sum a_1 \\
+&\frac{a_1, a_2, \ldots, a_6 \mid a_1, b_1, \ldots, f_1}{\Delta^2} \sum |a_1 b_2| \\
-&\frac{a_1, a_2, \ldots, a_6 \mid a_1, b_1, \ldots, f_1}{\Delta^1} \sum |a_1 b_2 c_3| \\
+&\frac{a_1, a_2, \ldots, a_6 \mid a_1, b_1, \ldots, f_1}{\Delta^0} \sum |a_1 b_2 c_3 d_4| \\
-&\qquad a_1 \sum |a_1 b_2 c_3 d_4 e_5| \\
+&\qquad |a_1 b_2 c_3 d_4 e_5 f_6| = 0.
\end{aligned}
$$

There is a particular appropriateness in writing Δ^4 in preference to Δ_4 or any other form, for it is an important property of a bipartite function, that if we substitute for its group of square arrays the single square array got by *multiplying* them together as if they were determinants, and retain the original initial and final lines, the new bipartite is equal to the old.

6. Let us now consider the quotient

$$
\begin{vmatrix} b_2-x & b_3 & b_4 \\ c_2 & c_3-x & c_4 \\ d_2 & d_3 & d_4-x \end{vmatrix} \div \begin{vmatrix} a_1-x & a_2 & a_3 & a_4 \\ b_1 & b_2-x & b_3 & b_4 \\ c_1 & c_2 & c_3-x & c_4 \\ d_1 & d_2 & d_3 & d_4-x \end{vmatrix}
$$

of which the dividend is the complementary minor of the element a_1-x of the divisor.

If both be arranged according to descending powers of x, and the division performed, the result must evidently be of the form

$$-x^{-1}-A_1x^{-2}-A_2x^{-3}-A_3x^{-4}-A_4x^{-5}-A_5x^{-6}-\ldots.$$

Multiplying this by the divisor, which, if we employ Σa_1, $\Sigma|a_1 b_2|$, ... with the signification above given to them, is

$$x^4-x^3\Sigma a_1+x^2\Sigma|a_1 b_2|-x\Sigma|a_1 b_2 c_3|+|a_1 b_2 c_3 d_4|,$$

or, for shortness, say,

$$x^4 - x^3\Sigma_1 + x^2\Sigma_2 - x\Sigma_3 + \Sigma_4,$$

we obtain

$$\begin{array}{r|r|r|r|r|r}
-x^3 - A_1 & x^2 - A_2 & x - A_3 & x^0 - A_4 & x^{-1} - A_5 & x^{-2} - \ldots \\
+\Sigma_1 & +A_1\Sigma_1 & +A_2\Sigma_1 & +A_3\Sigma_1 & +A_4\Sigma_1 & +\ldots \\
 & -\Sigma_2 & -A_1\Sigma_2 & -A_2\Sigma_2 & -A_3\Sigma_2 & -\ldots \\
 & & +\Sigma_3 & +A_1\Sigma_3 & +A_2\Sigma_3 & +\ldots \\
 & & & -\Sigma_4 & -A_1\Sigma_4 & -\ldots
\end{array}$$

which must consequently be equal to

$$-x^3 + (b_2 + c_3 + d_4)\,x^2 - (|b_2 c_3| + |b_2 d_4| + |c_3 d_4|)\,x + |b_2 c_3 d_4|.$$

The coefficients of like powers of x being equated, there results

$$\begin{aligned}
-A_1 + \Sigma_1 &= b_2 + c_3 + d_4, \\
-A_2 + A_1\Sigma_1 - \Sigma_2 &= -|b_2 c_3| - |b_2 d_4| - |c_3 d_4|, \\
-A_3 + A_2\Sigma_1 - A_1\Sigma_2 + \Sigma_3 &= |b_2 c_3 d_4|, \\
-A_4 + A_3\Sigma_1 - A_2\Sigma_2 + A_1\Sigma_3 - \Sigma_4 &= 0, \\
-A_5 + A_4\Sigma_1 - A_3\Sigma_2 + A_2\Sigma_3 - A_1\Sigma_4 &= 0, \\
\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots &\quad \ldots \quad \ldots \quad \ldots,
\end{aligned}$$

and hence, at once, by means of the identities (α), (α'), (α''), &c., we obtain

$$A_1 = a_1,$$

$$A_2 = \frac{a_1, a_2, a_3, a_4}{a_1, b_1, c_1, d_1},$$

$$A_3 = \frac{a_1 \ a_2 \ a_3 \ a_4}{\begin{array}{cccc|c} a_1 & b_1 & c_1 & d_1 & a_1 \\ a_2 & b_2 & c_2 & d_2 & b_1 \\ a_3 & b_3 & c_3 & d_3 & c_1 \\ a_4 & b_4 & c_4 & d_4 & d_1 \end{array}},$$

$$A_4 = \frac{a_1 \ a_2 \ a_3 \ a_4 \;\Big|\; a_1 \ b_1 \ c_1 \ d_1}{\begin{array}{cccc|cccc} a_1 & b_1 & c_1 & d_1 & a_1 & a_2 & a_3 & a_4 \\ a_2 & b_2 & c_2 & d_2 & b_1 & b_2 & b_3 & b_4 \\ a_3 & b_3 & c_3 & d_3 & c_1 & c_2 & c_3 & c_4 \\ a_4 & b_4 & c_4 & d_4 & d_1 & d_2 & d_3 & d_4 \end{array}},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

Consequently, we have the result

$$\begin{vmatrix} b_2-x & b_3 & b_4 \\ c_2 & c_3-x & c_4 \\ d_2 & d_3 & d_4-x \end{vmatrix} \div \begin{vmatrix} a_1-x & a_2 & a_3 & a_4 \\ b_1 & b_2-x & b_3 & b_4 \\ c_1 & c_2 & c_3-x & c_4 \\ d_1 & d_2 & d_3 & d_4-x \end{vmatrix}$$

$$= -x^{-1}-a_1x^{-2}-\frac{a_1,\, a_2,\, a_3,\, a_4}{a_1,\, b_1,\, c_1,\, d_1}\,x^{-3}-\frac{a_1\,a_2\,a_3\,a_4 \mid a_1\,b_1\,c_1\,d_1}{\Delta}\,x^{-4}$$

$$-\frac{a_1\,a_2\,a_3\,a_4 \mid a_1\,b_1\,c_1\,d_1}{\Delta^2}\,x^{-5}-\dots,$$

and may therefore look upon the quotient of the two determinants as the generating function of bipartites of the form

$$\frac{a_1\,a_2\,a_3\,a_4 \mid a_1\,b_1\,c_1\,d_1}{\Delta^m}.$$

7. A series of identities similar to (a), (a'), (a''), ..., viz.,

$$\frac{a_2,\, a_3,\, a_4}{b_1,\, c_1,\, d_1}-a_1\sum b_2=-|a_1\,b_2|-|a_1\,c_3|-|a_1\,d_4|,$$

$$\begin{array}{ccc|c} a_2 & a_3 & a_4 & \\ \hline b_2 & c_2 & d_2 & b_1 \\ b_3 & c_3 & d_3 & c_1 \\ b_4 & c_4 & d_4 & d_1 \end{array} -\frac{a_2,\, a_3,\, a_4}{b_1,\, c_1,\, d_1}\sum b_2+a_1\sum|b_2\,c_3|=\sum|a_1\,b_2\,c_3|,$$

$$\begin{array}{ccc|ccc} a_2 & a_3 & a_4 & b_1 & c_1 & d_1 \\ \hline b_2 & c_2 & d_2 & b_2 & b_3 & b_4 \\ b_3 & c_3 & d_3 & c_2 & c_3 & c_4 \\ b_4 & c_4 & d_4 & d_2 & d_3 & d_4 \end{array} - \begin{array}{ccc|c} a_2 & a_3 & a_4 & \\ \hline b_2 & c_2 & d_2 & b_1 \\ b_3 & c_3 & d_3 & c_1 \\ b_4 & c_4 & d_4 & d_1 \end{array} \sum b_2$$

$$+\frac{a_2\,a_3\,a_4}{b_1\,c_1\,d_1}\sum|b_2\,c_3|-a_1\,|b_2\,c_3\,d_4|=-|a_1\,b_2\,c_3\,d_4|,$$

... ...

leads in the same way to the result

$$\begin{vmatrix} a_1-x & b_1 & c_1 & d_1 \\ a_2 & b_2-x & c_2 & d_2 \\ a_3 & b_3 & c_3-x & d_3 \\ a_4 & b_4 & c_4 & d_4-x \end{vmatrix} \div \begin{vmatrix} b_2-x & c_2 & d_2 \\ b_3 & c_3-x & d_3 \\ b_4 & c_4 & d_4-x \end{vmatrix}$$

$$=-x+a_1+\frac{a_2\,a_3\,a_4}{b_1\,c_1\,d_1}\,x^{-1}+\frac{a_2\,a_3\,a_4 \mid b_1\,c_1\,d_1}{D}\,x^{-2}+\frac{a_2\,a_3\,a_4 \mid b_1\,c_1\,d_1}{D^2}\,x^{-3}+\dots,$$

where D stands for the square array of the determinant $|b_2\,c_3\,d_4|$.

8. The quotients of § 6 and § 7 being reciprocals, the multiplication of the two expansions ought to lead to a series of relations connecting coefficients of the one with coefficients of the other, that is to say, ought to give a series of identities regarding bipartites of the special kind we have been considering. As a matter of fact, what we obtain in this way is an expansion applicable to bipartites in general; the nature of it may be understood from the example

$$\begin{array}{ccc|ccc} a_1 & a_2 & a_3 & h_1 & k_1 & l_1 \\ \hline b_1 & c_1 & d_1 & e_1 & e_2 & e_3 \\ b_2 & c_2 & d_2 & f_1 & f_2 & f_3 \\ b_3 & c_3 & d_3 & g_1 & g_2 & g_3 \end{array}$$

$$= a_1 \cdot \frac{h_1 \;\; k_1 \;\; l_1}{\begin{array}{ccc|c} e_1 & e_2 & e_3 & b_1 \\ f_1 & f_2 & f_3 & b_2 \\ g_1 & g_2 & g_3 & b_3 \end{array}} + \frac{a_2\, a_3}{c_1\, d_1} \cdot \frac{h_1\, k_1\, l_1}{e_1\, e_2\, e_3} + \frac{a_2 \;\; a_3}{\begin{array}{cc|c} c_2 & d_2 & f_1 \\ c_3 & d_3 & g_1 \end{array}} \cdot h_1 + \frac{\begin{array}{cc|cc} a_2 & a_3 & k_1 & l_1 \end{array}}{\begin{array}{cc|cc} c_2 & d_2 & f_2 & f_3 \\ c_3 & d_3 & g_2 & g_3 \end{array}},$$

the first factors of the expansion being bipartites of the degrees 1, 2, 3, 4 and the second factors bipartites of the degrees 3, 2, 1, 0.

9. And now, as to Cayley's theorem in matrices. It asserts that "the determinant having for its matrix a given matrix, less the same matrix considered as a single quantity involving the matrix unity, is equal to zero;" for example, that if

$$M = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix},$$

then
$$\begin{vmatrix} a_1-M & a_2 & a_3 & a_4 \\ b_1 & b_2-M & b_3 & b_4 \\ c_1 & c_2 & c_3-M & c_4 \\ d_1 & d_2 & d_3 & d_4-M \end{vmatrix} = 0.$$

Expanding the determinant, we have

$$M^4 - M^3 \Sigma a_1 + M^2 \Sigma\, |a_1 b_2| - M^1 \Sigma\, |a_1 b_2 c_3| + M^0\, |a_1 b_2 c_3 d_4| = 0$$

as the identity to be established.

Now (§ 3),

$$M^4 = \begin{pmatrix} \dfrac{a_1, \ldots, a_4 \mid a_1, \ldots, d_1}{\Delta^2}, & \ldots\ldots, & \dfrac{a_1, \ldots, a_4 \mid a_4, \ldots, d_4}{\Delta^2} \\ \ldots & \ldots & \ldots \\ \dfrac{d_1, \ldots, d_4 \mid a_1, \ldots, d_1}{\Delta^2}, & \ldots\ldots, & \dfrac{d_1, \ldots, d_4 \mid a_4, \ldots, d_4}{\Delta^2} \end{pmatrix},$$

$$M^3 = \begin{pmatrix} \dfrac{a_1, \ldots, a_4 \mid a_1, \ldots, d_1}{\Delta}, & \ldots\ldots, & \dfrac{a_1, \ldots, a_4 \mid a_4, \ldots, d_4}{\Delta} \\ \ldots & \ldots & \ldots \\ \dfrac{d_1, \ldots, d_4 \mid a_1, \ldots, d_1}{\Delta}, & \ldots\ldots, & \dfrac{d_1, \ldots, d_4 \mid a_4, \ldots, d_4}{\Delta} \end{pmatrix},$$

$$M^2 = \begin{pmatrix} \dfrac{a_1, a_2, a_3, a_4}{a_1, b_1, c_1, d_1}, & \ldots\ldots, & \dfrac{a_1, a_2, a_3, a_4}{a_4, b_4, c_4, d_4} \\ \ldots & \ldots & \ldots \\ \dfrac{d_1, d_2, d_3, d_4}{a_1, b_1, c_1, d_1}, & \ldots\ldots, & \dfrac{d_1, d_2, d_3, d_4}{a_4, b_4, c_4, d_4} \end{pmatrix},$$

$$M^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and the zero on the right-hand side of the identity is *matrix* zero, *i.e.*,

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

We have thus sixteen algebraical identities to verify, viz.,

$$\frac{a_1, \ldots, a_4 \mid a_1, \ldots, d_1}{\Delta^2} - \frac{a_1, \ldots, a_4 \mid a_1, \ldots, d_1}{\Delta} \sum a_1$$
$$+ \frac{a_1, \ldots, a_4}{a_1, \ldots, d_1} \sum |a_1 b_2| - a_1 \sum |a_1 b_2 c_3| + |a_1 b_2 c_3 d_4| = 0,$$

and three others like it; and

$$\frac{a_1, \ldots, a_4 \mid a_2, \ldots, d_2}{\Delta^2} - \frac{a_1, \ldots, a_4 \mid a_2, \ldots, d_2}{\Delta} \sum a_1$$
$$+ \frac{a_1, \ldots, a_4}{a_2, \ldots, d_2} \sum |a_1 b_2| - a_2 \sum |a_1 b_2 c_3| + 0 = 0,$$

and eleven others like it. But these are exactly the identities obtained at the close of § 4. Hence Cayley's theorem is established.*

On the Potential of an Electrified Spherical Bowl, and on the Velocity Potential due to the Motion of an Infinite Liquid about such a Bowl.† *By* A. B. BASSET, M.A.

[*Read June* 11*th*, 1885.]

1. The object of this paper is to investigate the motion of an infinite liquid about a spherical bowl, which is either fixed in the liquid or moving in any manner whatever. The motion of the liquid is supposed to be caused by any system of sources, sinks, or vortex rings, the latter being so thin that the boundary conditions at their surfaces may be left out of account; and, for the same reason, moving solid bodies other than the bowl, are excluded from the causes by which the motion of the liquid is produced. It is shown that the velocity potential depends upon a certain function Ω, which is the magnetic potential of a complex magnetic shell of proper strength, which occupies the same position as the bowl. Now we know that the magnetic potential of such a shell can be deduced from a function V, which is the electro-static potential of a distribution of electricity upon the bowl, whose density at every point is equal to the strength of the shell at the same point. We are thus led, in the first place, to investigate the potentials of certain distributions of electricity upon the bowl, and I have therefore divided this paper into two parts, the first of which deals with electro-statics, and the second with hydrodynamics.

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* Mr. Glaisher drew my attention to previous proofs of Cayley's theorem, viz., BUCHHEIM, A., *Messenger of Math.*, XIII., pp. 65, 66; FORSYTH, A. R., *Messenger of Math.*, XIII., pp. 139—142. See also especially Sylvester's series of papers in the *Comptes Rendus* of the French Academy, the Johns Hopkins University *Circulars*, and the *American Jour. of Math.*, where, besides a draft of a proof, there is much matter (the theory of the latent roots of a matrix, &c.) closely connected with the subject, and where the much wider subject of matrix equations in general is entered upon. Other references which have been sent to me are—FROBENIUS, *Crelle's Journal*, LXXXIV.; WEYR, *Proc. of Bohemian Acad. of Sciences*.

† In consequence of the observations of one of the referees, a slight change in the title, and certain alterations in the text, which are indicated by square brackets, have been made.

So far as I am aware, only two papers have been published connected with the present subject. The first is the well-known investigation by Sir W. Thomson, of the distribution of electricity upon an uninfluenced spherical bowl;* the second is a paper by Dr. Ferrers,† in which he has not only obtained the potential of this distribution, but has also shown how to determine the potential and density of a bowl which is electrified in such a manner that the potential at its surface reduces to a zonal harmonic; and since the potential of every system of forces which is symmetrical about the axis of the bowl, can be expanded in a series of zonal harmonics, we have a method by means of which we can obtain the potential and density of a bowl which is under the influence of any such system of forces.

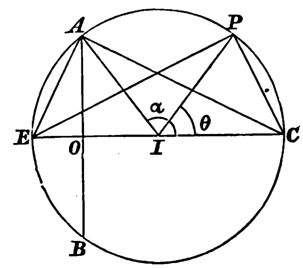
The first part commences with the determination of the density and potential of an uninfluenced plane screen of infinite extent, and having a circular aperture (which is a limiting form of a bowl); and the form of the result suggests the solution for a screen having an elliptic aperture. In the next place, the potential of a bowl under the action of a charge on the axis is obtained by inversion; and lastly, the potential and density of a bowl which is placed in a uniform field of parallel force are determined.

In the second part, the velocity potential of an infinite liquid in which a bowl is moving is determined, and also the velocity potential due to a source or sink situated on the axis. The corresponding results for a disc and screen are also noticed.

PART I.

2. *To find the potential and density of an uninfluenced electrified screen having a circular aperture.*

Let ACB be a section of the bowl through its axis, I the centre of the sphere of which it forms a part; let $AIC = \alpha$, $PIC = \theta$.



The total density upon an element of the bowl, including the charges on both sides of it, is

$$2m\left\{\sqrt{\left(\frac{1+\cos\alpha}{\cos\theta-\cos\alpha}\right)} - \tan^{-1}\sqrt{\left(\frac{1+\cos\alpha}{\cos\theta-\cos\alpha}\right)}+\frac{\pi}{2}\right\}\ldots(1),$$

* Reprint of papers on "Electro-Statics and Magnetism," p. 178.

† *Quarterly Journal*, Vol. XVIII., p. 97.

which is easily seen to be equal to

$$2m\left\{\frac{EA}{\sqrt{(EP^2-EA^2)}}+\cos^{-1}\frac{EA}{EP}\right\}.$$

If the centre of the bowl move off to infinity in the positive direction, the bowl becomes a screen, and E ultimately coincides with O, the centre of the aperture; hence the density of the screen on either side is

$$m\left\{\frac{c}{\sqrt{(r^2-c^2)}}+\cos^{-1}\frac{c}{r}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),*$$

where c is the radius of the aperture.

Take a line through O perpendicular to the screen as the axis of z, and let us find the potential at a point on this line whose distance from O is z.

Then $$V_z = 4\pi m\int_c^\infty\left\{\frac{c}{\sqrt{(r^2-c^2)}}+\cos^{-1}\frac{c}{r}\right\}\frac{r\,dr}{\sqrt{(r^2+z^2)}}.$$

Now $$\int_c^\infty \cos^{-1}\frac{c}{r}\cdot\frac{r\,dr}{\sqrt{(r^2+z^2)}}=\sqrt{(r^2+z^2)}\cos^{-1}\frac{c}{r}\Big|_c^\infty-\int_c^\infty\frac{c\sqrt{(r^2+z^2)}}{r\sqrt{(r^2-c^2)}}\,dr.$$

The term corresponding to the upper limit is infinite, but constant, and may therefore be left out of account; hence

$$V_z = -4\pi mcz^2\int_c^\infty\frac{dr}{r\sqrt{(r^2-c^2)(r^2+z^2)}}$$

$$= 4\pi mz\left(-\frac{\pi}{2}+\tan^{-1}\cancel{\frac{z}{c}}\right).\ \frac{c}{z}$$

Hence the general value of V at the point (r,θ) is

$$V = 4\pi m\left\{c-\frac{\pi z}{2}+c\Sigma_1^\infty\frac{(-)^n P_{2n-1}}{2n+1}\left(\frac{c}{r}\right)^{2n}\right\}\quad [r>c]\ \ldots(3),$$

and $$V = 4\pi mc\Sigma_1^\infty\frac{(-)^n P_{2n}}{2n-1}\left(\frac{r}{c}\right)^{2n}\quad [r<c]\ \ldots\ldots\ldots\ldots\ldots(4),$$

where P_n is the zonal harmonic of degree n.

To sum the series (3), we have

$$2\left(\frac{P_1h^3}{3}+\frac{P_3h^5}{5}+\frac{P_5h^7}{7}+\ldots\right)=\int_0^h\frac{h\,dh}{\sqrt{(1-2h\mu+h^2)}}-\int_0^h\frac{h\,dh}{\sqrt{(1+2h\mu+h^2)}}$$

$$=\sqrt{(1-2h\mu+h^2)}-\sqrt{(1+2h\mu+h^2)}+\mu\log\frac{\mu+h+\sqrt{(1+2h\mu+h^2)}}{\mu-h+\sqrt{(1-2h\mu+h^2)}},$$

---

* This result is given in Watson and Burbury's "Electricity and Magnetism," p. 139, which has appeared since the above was written.

putting $h = c\iota/r$, where $\iota = \sqrt{(-1)}$, we have

$$-2\iota\left(\frac{c^3P_1}{3r^2}-\frac{c^5P_3}{5r^4}+\quad\right)=(r^2-2\iota cr\mu-c^2)^{\frac{1}{2}}-(r^2+2\iota cr\mu-c^2)^{\frac{1}{2}}$$

$$+r\mu\log\frac{r\mu+c\iota+\sqrt{(r^2+2\iota cr\mu-c^2)}}{r\mu-c\iota+\sqrt{(r^2-2\iota cr\mu-c^2)}}.$$

Let
$$r^2-c^2=\lambda\cos 2a,$$
$$2cr\mu=\lambda\sin 2a,$$

so that
$$\left.\begin{aligned}\lambda&=\sqrt{(r^4+c^4+2r^2c^2\cos 2\theta)}\\ \cos a&=\sqrt{\left(\frac{\lambda+r^2-c^2}{2\lambda}\right)}\\ \sin a&=\sqrt{\left(\frac{\lambda-r^2+c^2}{2\lambda}\right)}\end{aligned}\right\}\ \dots\dots\dots\dots\dots\dots(5).$$

Then the series

$$=-2\iota\sqrt{\lambda}\sin a+2\iota z\tan^{-1}\frac{c+\sqrt{\lambda}\sin a}{r\cos\theta+\sqrt{\lambda}\cos a}$$

$$=-2\iota\sqrt{\lambda}\sin a+2\iota z\tan^{-1}\frac{c}{\sqrt{\lambda}\sin a},$$

therefore

$$V=4\pi m\left\{c-\sqrt{\left(\frac{\lambda-r^2+c^2}{2}\right)}\mp z\tan^{-1}\frac{\sqrt{(\lambda+r^2-c^2)}}{c\sqrt{2}}\right\}\ \dots\dots(6);$$

the upper or lower sign being taken according as the point (r, θ) lies on the positive or negative side of the screen.

We have supposed that $r>c$; but, if we had supposed $r<c$ and summed the second series (4), we should have obtained the same value for V.

To test this result: when $\theta = \frac{1}{2}\pi$ and $r>c$, $\lambda = r^2-c^2$, therefore at the screen
$$V = 4\pi mc\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(7);$$

when $\theta = \frac{1}{2}\pi$ and $r<c$, $\lambda = c^2-r^2$, therefore at the aperture

$$V = 4\pi m\left\{c-\sqrt{(c^2-r^2)}\right\}\ \dots\dots\dots\dots\dots\dots(8).$$

3. We may express this result in a more elegant form as follows. Let P be the point at which the potential is sought, and let the plane through OP, which is perpendicular to the screen, cut the boundary of the aperture in A and B, then

$$AP = (r^2+c^2-2cr\cos\theta)^{\frac{1}{2}} = p,$$

$$BP = (r^2+c^2+2cr\cos\theta)^{\frac{1}{2}} = q;$$

and
$$\lambda - r^2 + c^2 = \tfrac{1}{2}\{(p+q)^2 - 4r^2\},$$
$$\lambda + r^2 - c^2 = \tfrac{1}{2}\{(p+q)^2 - 4c^2\},$$
therefore
$$V = 4\pi m\left\{c - \tfrac{1}{2}\sqrt{(p+q+2r)(p+q-2r)} \mp r\cos\theta\cos^{-1}\frac{2c}{p+q}\right\}\ \ldots(9).$$

4. *To find the potential of an uninfluenced screen having an elliptic aperture.*

Returning to (6), let us take any two lines through O in the plane of the screen as the axes of x and y, and let μ and ν be elliptic co-ordinates determined by the equations
$$\frac{x^2+y^2}{1-\mu^2} - \frac{z^2}{\mu^2} = c^2,$$
$$\frac{x^2+y^2}{1+\nu^2} + \frac{z^2}{\nu^2} = c^2,$$
so that
$$\sqrt{(x^2+y^2)} = c\sqrt{(1+\nu^2)(1-\mu^2)},$$
$$z = c\mu\nu,$$
and
$$\lambda - r^2 + c^2 = 2c^2\mu^2,$$
$$\lambda + r^2 - c^2 = 2c^2\nu^2.$$

Then V can be expressed in the form
$$V = 4\pi mc\left\{1 - P_1(\mu)\,q_1(\nu) - \frac{\pi}{2}P_1(\mu)\,p_1(\nu)\right\}\ \ldots\ldots\ldots(10),$$
where $P_1(\mu)$ is a zonal harmonic, and the p and q functions are the particular kinds of spheroidal harmonics which are employed in discussing problems relating to the potentials of planetary ellipsoids.

The form of this result at once suggests the solution when the aperture is an ellipse instead of a circle. Consider the expression
$$V = 4\pi m\sqrt{(ab)} \mp 2\pi mab^2 z\int_\lambda^\infty \frac{d\phi}{\sqrt{(a^2+\phi)}\sqrt{(b^2+\phi)}\,\phi^{\frac{3}{2}}} \mp 4\pi mzE\ldots(11),$$
where E is the complete elliptic integral of the second kind to mod $\sqrt{(a^2-b^2)}\,/\,a$, λ is the positive root of the equation
$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{\lambda} = 1,$$
and the upper or lower sign is to be taken according as the point (x, y, z) lies on the positive or negative side of the screen. The axes

of x and y are the directions of the major and minor axes of the elliptic aperture.

We know from the Theory of Attractions that V satisfies Laplace's equation, and that V and its first derivatives are continuous at all points of space which do not lie on the plane $z = 0$. Let

$$\operatorname{sn}\psi = \frac{a}{\sqrt{(a^2+\phi)}},$$

$$\operatorname{sn}\alpha = \frac{a}{\sqrt{(a^2+\lambda)}};$$

then the definite integral

$$= \frac{2}{a^3}\int_0^{\alpha}\frac{\operatorname{sn}^2\psi}{\operatorname{cn}^2\psi}\,d\psi$$

$$= \frac{2}{a^3k'^2}\left\{\frac{\operatorname{sn}\alpha\operatorname{dn}\alpha}{\operatorname{cn}\alpha} - Z(\alpha) - \alpha\frac{E}{K}\right\},$$

and $$\frac{V}{4\pi m} = \sqrt{(ab)} \mp z\left\{a\sqrt{\left(\frac{b^2+\lambda}{\lambda(a^2+\lambda)}\right)} - Z(\alpha) - \alpha\frac{E}{K}\right\} \mp zE.$$

At points on the screen, $z = 0$ and λ is not zero, therefore

$$V = 4\pi m\sqrt{(ab)} \qquad\qquad (12)$$

At points on the aperture, $z = 0$, $\lambda = 0$, $\alpha = K$, and

$$\frac{z}{\sqrt{\lambda}} = \sqrt{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)};$$

therefore $$V = 4\pi m\left\{\sqrt{(ab)} - b\sqrt{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)}\right\} \qquad (13).$$

Also $$\mp\frac{dV}{dz} = 4\pi m\left\{a\sqrt{\left(\frac{b^2+\lambda}{\lambda(a^2+\lambda)}\right)} - Z(\alpha) - \alpha\frac{E}{K} + E - \frac{ab^2z^2}{\lambda^{\frac{3}{2}}(a^2+\lambda)^{\frac{1}{2}}(b^2+\lambda)^{\frac{1}{2}}\left\{\frac{x^2}{(a^2+\lambda)^2} + \frac{y^2}{(b^2+\lambda)^2} + \frac{z^2}{\lambda^2}\right\}}\right\}.$$

Therefore at the aperture

$$\mp\frac{dV}{dz} = 4\pi m\left(\frac{b}{\sqrt{\lambda}} - \frac{b}{\sqrt{\lambda}}\right) = 0.$$

At the screen dV/dz does not vanish but is discontinuous; hence there is a surface density, which

$$= m\left\{a\sqrt{\left(\frac{b^2+\lambda}{\lambda(a^2+\lambda)}\right)} - Z(\alpha) - \alpha\frac{E}{K}\right\} + mE \qquad (14).$$

5. *To find the potential of a bowl under the action of a charge on the axis.*

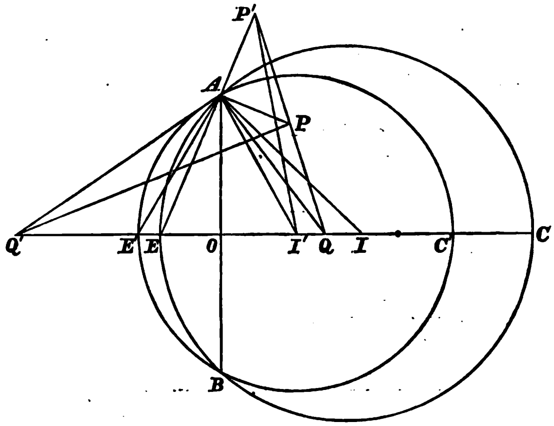


Fig. 2.

Let $AC'B$ be the bowl, I' its centre, Q a charge on its axis. Invert with respect to Q, and take the constant of inversion equal to QA; let ACB be the inverted bowl, I its centre, Q' the inverse point of I'.

Then, since $$QI'.QQ' = QA^2,$$

therefore $$I'AQ = AQ'Q;$$

also, since $$QE'.QE = QA^2,$$

$$QAE = QE'A = I'AE';$$

therefore $$E'AE = I'AQ,$$

therefore $$IAQ = IAE - QAE = QEA - QE'A$$
$$= E'AE = I'AQ = AQ'Q,$$

therefore $$IQ'.IQ = IA^2;$$

therefore Q' is the image of Q with respect to the bowl ACB.

Now, if the bowl $AC'B$ be uninfluenced, its potential at all points external to itself and the sphere passing through its centre and its rim, is*

$$V' = m'\left\{\frac{\pi}{I'P'} + \frac{1}{I'A}\sin^{-1}\frac{AB}{AP'+BP'} - \frac{1}{I'P'}\sin^{-1}\frac{I'P'}{I'A}\cdot\frac{AB}{AP'+BP'}\right\}.$$

* *Quarterly Journal*, Vol. XVIII., p. 106.

Now $$\frac{AP'}{QA} = \frac{AP}{QP},$$

therefore $$AP'+BP' = \frac{QA}{QP}(AP+BP).$$

Also $$\frac{I'P'}{QP'} = \frac{Q'P}{QQ'} = \frac{QA^2}{QQ'.QP},$$

therefore $$\frac{I'P'}{I'A} = \frac{Q'P.QA}{QP.Q'A},$$

therefore $$V = \frac{m}{\pi Q'A}\left\{\frac{\pi Q'A}{Q'P} + \frac{QA}{QP}\sin^{-1}\frac{QP.AB}{QA(AP+BP)} - \frac{Q'A}{Q'P}\sin^{-1}\frac{Q'P.AB}{Q'A(AP+BP)}\right\} \quad \ldots\ldots(\alpha).$$

This is the potential of the bowl when under the action of a charge $-m$ at Q, at all points within the space bounded by the bowl and a sphere passing through Q' and its rim.

Now, suppose a point to start from the interior surface of the bowl, and to move round to the outside without passing through the bowl. When the point arrives at the last mentioned sphere,

$$\sin^{-1} Q'P.AB / Q'A(AP+BP)$$

becomes a right angle; hence, after we have crossed this sphere, we must write $\pi - \sin^{-1}$ for $\sin^{-1}$, and the value of V becomes

$$V = \frac{m}{\pi Q'A}\left\{\frac{Q'A}{Q'P}\sin^{-1}\frac{Q'P.AB}{Q'A(AP+BP)} + \frac{QA}{QP}\sin^{-1}\frac{QP.AB}{QA(AP+BP)}\right\} \quad \ldots\ldots\ldots\ldots\ldots(\beta).$$

As soon as the point crosses that portion of the sphere through Q, and the rim of the bowl which lies outside the sphere of which the bowl forms a part, $\sin^{-1} QP.AB / QA(AP+BP)$ becomes a right angle; hence, *outside* the bowl and the above-mentioned sphere,

$$V = \frac{m}{\pi Q'A}\left\{\frac{QA}{QP} - \frac{QA}{QP}\sin^{-1}\frac{QP.AB}{QA(AP+BP)} + \frac{Q'A}{Q'P}\sin^{-1}\frac{Q'P.AB}{Q'A(AP+BP)}\right\} \quad \ldots\ldots\ldots\ldots\ldots(\gamma).$$

The value of V for the remaining portion of space is given by (β).

The corresponding results for a disc or a screen are of the same form as those which we have just obtained for a bowl, and must be interpreted in a similar manner.

6. *To find the potential when the bowl is placed in a uniform field of force parallel to its axis.*

The simplest way of solving this problem is to utilise the result of the last article, by placing a positive and a negative charge of strength m, at two points on the axis at distances f from the centre, and on opposite sides of it, and making them move off to infinity, whilst the product m/f^2 remains constant.

The potential due to the positive charge at Q (Q being an external point) is

$$V' = \frac{m}{\pi}\left[\frac{1}{QP}\sin^{-1}\frac{QP\,.\,\lambda}{QA} + \frac{Q'A}{QA\,.\,Q'P}\sin^{-1}\frac{Q'P\,.\,\lambda}{Q'A}\right],$$

where $$\lambda = \frac{AB}{AP+BP}.$$

Putting $u = 1/f$, and remembering that we need not retain powers of u higher than the first, the value of V' may be written

$$\begin{aligned} V' = \frac{m}{\pi f}\Big[&(1+ru\cos\theta)\sin^{-1}\lambda\,\{1+u\,(a\cos\alpha - r\cos\theta)\} \\ &+\frac{a}{r}\left(1+\frac{a^2}{r}\,a\cos\theta\right)\sin^{-1}\frac{\lambda r}{a}\left\{1+u\left(a\cos\alpha-\frac{a^2}{r}\cos\theta\right)\right\}\Big] \\ = \frac{m}{\pi f}&\{F(u)+\phi(u)\} \text{ say.}\end{aligned}$$

If, therefore, we put $2m/f^2 = -\dfrac{Ca}{2}$, the potential of the bowl, when placed in a field of force whose potential is $\frac{1}{2}Caz$, will be

$$= -\frac{Ca}{2\pi}\{F'(0)+\phi'(0)\}.$$

Now,

$$F'(0) = r\cos\theta\sin^{-1}\lambda + \frac{\lambda\,(a\cos\alpha - r\cos\theta)}{(1-\lambda^2)^{\frac{1}{2}}},$$

and, if we put $AP = q$, $BP = p$, then (see Fig. 3)

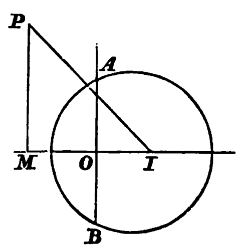


FIG. 3.

$$\begin{aligned} OM^2 &= (a\cos\alpha - r\cos\theta)^2 \\ &= q^2-(PM-c)^2 \\ &= p^2-(PM+c)^2, \end{aligned}$$

therefore $$OM^2 = \frac{(p^2+q^2)\,8c^2 - 16c^4 - (p^2-q^2)^2}{16c^2}$$

$$= \frac{\{(p+q)^2-4c^2\}\,\{4c^2-(p-q)^2\}}{16c^2},$$

therefore $\quad F'(0) = r\cos\theta\sin^{-1}\dfrac{2c}{p+q} + \tfrac{1}{2}\{4c^2-(p-q)^2\}^{\frac{1}{2}}.$

The value of $\phi'(0)$ can be obtained from that of $F'(0)$, by changing r into a^2/r, and multiplying the result by a/r; hence

$$\phi'(0) = \frac{a^3}{r^2}\cos\theta\sin^{-1}\frac{2r\sin\alpha}{p+q} + \frac{a^3}{2r^3}\{4r^2\sin^2\alpha-(p-q)^2\}^{\frac{1}{2}},$$

and the value of V may be written

negative

$$V = -\frac{Ca}{2\pi}\Big[r\cos\theta\sin^{-1}\frac{2c}{p+q} \pm \tfrac{1}{2}\{4c^2-(p-q)^2\}^{\frac{1}{2}}$$
$$+\frac{a^3}{r^2}\cos\theta\sin^{-1}\frac{2r\sin\alpha}{p+q} \pm \frac{a^3}{2r^3}\{4r^2\sin^2\alpha-(p-q)^2\}^{\frac{1}{2}}\Big] \quad\text{...(15).}$$

positive

From the latter part of the preceding article, it appears that, if the positive signs be taken, this is the potential at all points within the space bounded by the plane passing through the rim of the bowl, and that portion of the sphere passing through the centre and rim of the bowl, which lies outside the bowl.

The potential for the space enclosed by the bowl and the plane through its rim, is obtained by changing the inverse sine in the first term to $\pi-\sin^{-1}$, and placing the negative sign before the second term.

The potential of the remaining portion of space is obtained by changing the inverse sine in the third term to $\pi-\sin^{-1}$, and placing the negative sign before the fourth term.

Hence we see that at the surface of the bowl, V is always equal to $-\frac{1}{2}Caz$; its value at the unoccupied portion of the sphere is

$$-\frac{Ca^2}{\pi}\left\{\sqrt{(1-\cos\alpha)(\cos\alpha-\cos\theta)} + \cos\theta\tan^{-1}\sqrt{\left(\frac{1-\cos\alpha}{\cos\alpha-\cos\theta}\right)}\right\} \quad\text{...............(16).}$$

The density of the bowl on the convex side is

$$\sigma = -\frac{Ca}{8\pi^2}\left\{3\cos\theta\cot^{-1}\sqrt{\left(\frac{1+\cos\alpha}{\cos\theta-\cos\alpha}\right)} + (3\cos\theta-1-\cos\alpha)\sqrt{\left(\frac{1+\cos\alpha}{\cos\theta-\cos\alpha}\right)}\right\} \quad\text{...(17),}$$

$$-\frac{Ca\cos\theta}{8\pi},$$

and on the concave side it

$$= \sigma + \frac{Ca\cos\theta}{8\pi} \quad\text{.............................(18).}$$

[7. *To find the potential of a bowl under the action of a charge situated in the plane containing the rim of the bowl.*

As we only have occasion to employ this result for the purpose of obtaining the potential when the bowl is placed in a field of force which is perpendicular to a plane passing through the axis, it will be sufficient to consider the case of an external point.

Let $CA'E$ (Fig. 4) be the plane passing through the axis of the bowl and any point P'; and let us take k the constant of inversion equal to the tangent from Q, the influencing point, to the sphere; the bowl will then invert into itself.

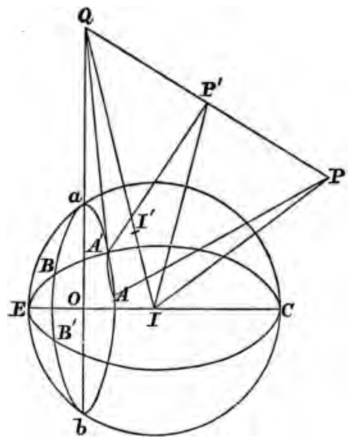


Fig. 4.

Now $$\frac{A'P'}{QA'} = \frac{AP}{QP},$$

and $$\frac{B'P'}{QB'} = \frac{BP}{QP}.$$

Also

$$IP'^2 = QI^2 + QP'^2 - 2QI.QP'\cos IQP'$$

$$= QI^2 + \frac{k^4}{QP^2} - \frac{2QIk^2}{QP}\cos IQP',$$

and $$IP^2 = r^2 = QI^2 + QP^2 - 2QI.QP\cos IQP.$$

Putting $QO = h$, so that $k^2 = h^2 - c^2$, and eliminating $\cos IQP$, we find

$$IP'^2 = \frac{(r^2-a^2)(h^2-c^2) + QP^2a^2}{QP^2} \quad\text{.....................(19).}$$

Hence $$V = \frac{m}{\pi}\left[\frac{1}{QP}\sin^{-1}\frac{2c.QP}{AP.QA'+BP.QB'} + \frac{a}{QP.IP'}\sin^{-1}\frac{2c.IP'.QP}{a(AP.QA'+BP.QB')}\right].$$

[8. *To find the potential when the bowl is placed in a field of force perpendicular to a plane passing through the axis.*

Applying the method of Art. 6, and observing that AO and OB will ultimately be in the same straight line, we have (Fig. 5)

$$\cos\phi = \frac{\cos QIP + \cos\beta\cos\theta}{\sin\theta\sin\beta},$$

therefore

$$QI\cos QIP = h\cos\phi\sin\theta - OI\cos\theta,$$

and $QP^2 = QI^2 + r^2 - 2QIr\cos QIP$,

therefore $QP = h\,(1 - ru\cos\phi\sin\theta)$,

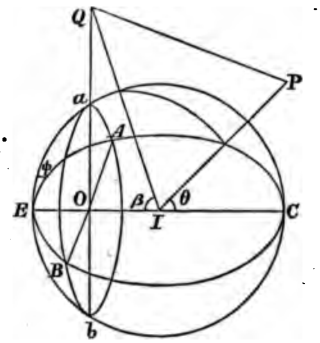


Fig. 5.

if $u = 1/h$, and the second and higher powers of u are neglected.

Similarly, by (19),

$$IP'^2 \,.\, QP^2 = h^2\,(r^2 - 2a^2 ru\cos\phi\sin\theta),$$

therefore $$IP' \,.\, QP = hr\left(1 - \frac{a^2}{r}u\cos\phi\sin\theta\right),$$

therefore

$$V = \frac{m}{\pi h}\left\{(1 + ru\cos\phi\sin\theta)\sin^{-1}\frac{2c\,(1 - ru\cos\phi\sin\theta)}{AP + BP - (BP - AP)\,cu\cos\phi}\right.$$

$$\left. + \frac{a}{r}\left(1 + \frac{a^2}{r}u\cos\phi\sin\theta\right)\sin^{-1}\frac{2cr\left(1 - \frac{a^2}{r}u\cos\phi\sin\theta\right)}{a\,\{AP + BP - (BP - AP)\,cu\cos\phi\}}\right\}$$

$$= \frac{m}{\pi h}\{F(u) + \phi(u)\} \text{ say.}$$

If, therefore, we put $2m/h^2 = -\dfrac{Aa}{2}$, the potential of the bowl, when placed in a field of force whose potential is $\dfrac{Aa}{2}\sin\theta\cos\phi$, will be

$$= -\frac{Aa}{2\pi}\{F'(0) + \phi'(0)\}.$$

Now $$F'(0) = r\cos\phi\sin\theta\left[\sin^{-1}\frac{2c}{p+q} - \frac{2c}{(p+q)^2}\{(p+q)^2 - 4c^2\}^{\frac{1}{2}}\right],$$

where $AP = p$ and $BP = q$. Also $\phi'(0)$ is obtained from $F'(0)$ by changing r into a^2/r, and multiplying the result by a/r. Therefore

$$\phi'(0) = \frac{a^3}{r^3}\cos\phi\sin\theta\left[\sin^{-1}\frac{2r\sin\alpha}{p+q} - \frac{2r\sin\alpha}{(p+q)^2}\{(p+q)^2 - 4r^2\sin^2\alpha\}^{\frac{1}{2}}\right].$$

and the value of V may be written

$$V = -\frac{Aa}{2\pi}\cos\phi\sin\theta\left[r\sin^{-1}\frac{2c}{p+q} \mp \frac{2cr}{(p+q)^2}\{(p+q)^2-4c^2\}^{\frac{1}{2}}\right.$$

$$\left. + \frac{a^3}{r^3}\sin^{-1}\frac{2r\sin a}{p+q} \mp \frac{2a^3\sin a}{r(p+q)^2}\{(p+q)^2-4r^2\sin^2 a\}^{\frac{1}{2}}\right] \ldots\ldots (20).$$

The inverse sines, and the double signs before the second and fourth terms, must be interpreted in the manner explained in Art. 6; hence it appears that, at the surface of the bowl, V will be equal to

$$-\tfrac{1}{2}Aa\cos\phi\sin\theta,$$

and that its value at the unoccupied portion of the sphere is

$$-\frac{Aa^2}{\pi}\cos\phi\sin\theta\left[\tan^{-1}\sqrt{\left(\frac{1-\cos a}{\cos a-\cos\theta}\right)}\right.$$

$$\left. -\frac{1}{\sqrt{2}}\operatorname{cosec}^2\frac{\theta}{2}\sin\frac{a}{2}(\cos a-\cos\theta)^{\frac{1}{2}}\right] \ldots (21).]$$

[9. It has been stated in the introduction, that the problem of determining the potential of a spherical bowl under the action of any system of forces which is symmetrical about the axis, has been solved by Dr. Ferrers. In order to apply a similar method, to determine the potential when the bowl is under the action of an unsymmetrical system of forces, it would be necessary to obtain formulæ giving the potential and density when the bowl is under the action of a system of forces, whose potential at the surface of the bowl is equal to the tesseral harmonic $\sin(m\phi+e_m)\,P_n^m(\cos\theta)$. I regret that I have not as yet been able to solve this problem; but at the same time another, although indirect, method exists of dealing with this question. It is known that the potential at an external point of any distribution of electricity, symmetrical or otherwise, upon a planetary ellipsoid, and therefore upon a circular disc, can be expressed in a series of spheroidal harmonics. If, therefore, we can effect the expansion of the potential of the influencing system, in a suitable series of spheroidal harmonics, we can obtain the potential of the disc, and proceed to the case of a bowl by inversion. The application of this method is, however, not free from difficulty, since the expansion of functions in series of spheroidal harmonics, and the summation of such series, are operations not always easy to effect.

A third method would be to obtain the potential of a disc by means of definite integrals involving Bessel's functions. This can be effected without much difficulty in the case of a symmetrical potential, but I

am not certain whether the method admits of easy extension to unsymmetrical potentials.]

PART II.

10. Let us suppose that the motion of an infinite liquid is caused by any system of sources, sinks, or vortex filaments; let Φ be velocity potential due to this system (which we shall call the external system) when the bowl is absent; and let ϕ be the velocity potential after the bowl has been introduced. Then we may put

$$\phi = \Omega + \Phi,$$

where Ω is to be determined.

If the bowl is fixed, which for the present we shall suppose to be the case, the only surface condition is

$$-\frac{d\Omega}{dr} = \frac{d\Phi}{dr},$$

when $r = a$. This condition is to be satisfied on both sides of the bowl.

Now, if we remove the bowl, and substitute over its surface a sheet of doublets, whose axes are in the directions of the radii passing through them, and whose strength σ is such that the normal velocity due to them at any point of the bowl is equal and opposite to the normal velocity due to Φ, all the conditions of the problem will be satisfied. But the velocity potential of such a sheet of doublets is analytically equivalent to the magnetic potential of a complex magnetic shell of the same strength, which occupies the position of the bowl, and whose positive side coincides with the sink side of the sheet of doublets; hence the problem is reduced to finding the potential and strength of such a magnetic shell when the normal component of the magnetic force at the surface of the shell is given.

Now we know that, if V be the potential of a surface distribution of matter upon the bowl of density σ, then

$$\Omega = -\frac{1}{a}\frac{d(Vr)}{dr},$$

and that, if Ω_0 and Ω_i be the values of Ω at two contiguous points just outside and just inside the shell respectively, then

$$\Omega_0 - \Omega_i = 4\pi\sigma.$$

The magnetic force at the surface of the bowl is

$$-\frac{d\Omega}{dr}=\frac{1}{a}\frac{d^2(Vr)}{dr^2}$$

$$=-\frac{1}{a^2}\left\{\frac{d}{d\mu}(1-\mu^2)\frac{dV}{d\mu}+\frac{1}{1-\mu}\frac{d^2V}{d\psi^2}\right\}.$$

Now the value of the magnetic force at the surface of the bowl can always be expanded in a series of spherical harmonics; hence, if

$$-\frac{d\Omega}{\delta r}=Y_n,$$

therefore $$V=\frac{a^2Y_n}{n(n+1)}.$$

Hence, if $$-\frac{d\Omega}{dr}=\Sigma_1^\infty Y_n \quad\text{.............................(22)}$$

at the surface, the corresponding value of V is

$$V=a^2\Sigma_1^\infty\frac{Y_n}{n(n+1)} \quad\text{..........................(23).}$$

The formula (23) fails when $n=0$; the only case, however, which is necessary for our purpose to consider, is when the magnetic force is symmetrical with respect to the axis of the bowl, and has a constant value F at its surface. In this case,

$$F=-\frac{d\Omega}{dr}$$

$$=-\frac{1}{a^2}\frac{d}{d\mu}(1-\mu^2)\frac{dV}{d\mu},$$

therefore $$V=\tfrac{1}{2}Fa^2\log(1-\mu^2)+\tfrac{1}{2}A\log\frac{1+\mu}{1-\mu}+B.$$

Now V must not be infinite when $\mu=1$, therefore

$$A=Fa^2,$$

and the value of V may be written

$$V=Fa^2\log a(1+\mu).$$

But, if an infinite straight line extending from the centre of the bowl to $-\infty$, be electrified with line density Fa^2, its potential is

$$=-Fa^2\log r(1+\mu).$$

Hence V is the potential of the induced charge when the bowl is under the action of a positively electrified line extending from the centre to $-\infty$. If, therefore, χ be the potential of the bowl, under the action of a positive charge of unit intensity, situated at a point on the axis distant u from the centre, and in the negative side of it,

$$V = Fa^2 \int_0^\infty \chi \, du,$$

where the value of χ is obtained by means of the formulæ (α), (β), (γ) of Art. 5.

11. The preceding result enables us to find the velocity potential due to a source situated at the centre of the bowl. In this case

$$\Phi = -\frac{m}{r},$$

therefore $$-\frac{d\Omega}{dr} = \frac{m}{a^2},$$

therefore $$\phi = -\frac{m}{a}\int_0^\infty \frac{d(r\chi)}{dr}\,du - \frac{m}{r} \quad\text{.....................(24).}$$

12. *To find the velocity potential due to a source placed anywhere on the axis of the bowl.*

1st. Let the source be placed *outside* the bowl on the positive side of the centre, and at a distance f from it.

Then $$\Phi = -\frac{m}{(f^2+r^2-2rf\cos\theta)^{\frac{1}{2}}}$$

$$= -\frac{m}{f}\Sigma P_n\left(\frac{r}{f}\right)^n.$$

Therefore at the surface

$$\frac{d\Omega}{dr} = \frac{m}{f^2}\Sigma_1^\infty nP_n\left(\frac{a}{f}\right)^{n-1},$$

therefore $$V = -m\Sigma_1^\infty \frac{P_n}{n+1}\left(\frac{a}{f}\right)^{n+1}$$

$$= -m\int_0^{a^2/f}\frac{du}{(u^2-2au\mu+a^2)^{\frac{1}{2}}} + \frac{ma}{f} \quad\text{...............(25).}$$

V is therefore the potential of the induced charge when the bowl is under the action of a system of forces whose potential

$$U = m\int_0^{a^2/f}\frac{du}{(u^2-2ru\mu+r^2)^{\frac{1}{2}}} - \frac{ma^2}{fr} \quad\text{...............(26).}$$

The first term is the potential of a positively electrified line extending from the image of the source to the centre of the bowl. The second term is the potential of a charge $-ma^2/f$ at the centre; hence, if χ_1 is the potential due to a positive charge of unit intensity distant u from the centre,

$$V = m\int_0^{a^2/f} \chi_1\, du + \frac{ma^2}{f}\chi_0 \quad\text{.....................} \quad (27),$$

and

$$\phi = -\frac{m}{a}\int_0^{a^2/f} \frac{d\,(r\chi_1)}{dr}\, du - \frac{ma}{f}\frac{d\,(r\chi_0)}{dr} + \Phi \quad\text{.........} \quad (28).*$$

2nd. Let the source be placed *inside* the bowl on the positive side of the centre, and at a distance f from it.

Then

$$\Phi = -\frac{m}{(f^2+r^2-2rf\cos\theta)^{\frac{1}{2}}}$$

$$= -\frac{m}{r}\,\Sigma P_n\left(\frac{f}{r}\right)^n.$$

Therefore at the surface

$$-\frac{d\Omega}{dr} = \frac{m}{a^2} + \frac{m}{a^2}\Sigma_1^\infty\,(n+1)\,P_n\left(\frac{f}{a}\right)^n \quad\text{..............} \quad (29).$$

The last term will give rise to a term in V which

$$= m\,\Sigma_1^\infty \frac{P_n}{n}\left(\frac{f}{a}\right)^n \quad\text{...} \quad (29a).$$

$$= m\int_{a^2/f}^\infty \frac{du}{(u^2+a^2-2au\mu)^{\frac{1}{2}}} - m\int_{a^2/f}^\infty \frac{du}{u} \quad\text{.................} \quad (30).$$

This portion of V is therefore the potential of the induced charge when the bowl is under the influence of a system of forces whose potential is

$$-m\int_{a^2/f}^\infty \frac{du}{(u^2+r^2-2ru\mu)^{\frac{1}{2}}} + m\int_{a^2/f}^\infty \frac{du}{u}.$$

Both these integrals are infinite, but their difference is finite. The second one is an infinite constant, and may therefore be disregarded. The first one is the potential of an infinite straight line negatively electrified with line density m, which extends from the image of the

* In performing the integration the second term will disappear, as it will be cancelled by a corresponding quantity in the first term.

source to infinity. This portion of V is therefore the potential of the induced charge when the bowl is under the influence of such a line.

The first term of (29) gives rise to a term in V, which, by Art. 10, is the potential of the bowl when under the influence of a positively electrified line extending from the centre to $-\infty$.

We may therefore write

$$V = m\int_0^\infty \chi\, du - m\int_{a^2/f}^\infty \chi_1\, du \qquad\text{.....................(31)},$$

with the understanding that infinite constant terms which appear in the integration are to be disregarded; hence the value of ϕ is

$$\phi = -\frac{m}{a}\frac{d(Vr)}{dr} + \Phi \qquad\text{..........................(32)},$$

where the value of V is given by (31).

From the above investigation it appears that, if the source be situated on the negative side of the centre, V will be the potential when the bowl is under the influence of a positively electrified line extending from the centre to the image of the source.

13. A different investigation of the last result may be sketched out as follows:—The general value of (29a) outside the bowl is, by the theorem of Dr. Ferrers, referred to in the introduction,

$$V = \frac{m}{\pi}\Sigma_{n=1}^{n=\infty}\frac{1}{n}\left(\frac{f}{a}\right)^n \Sigma_{s=0}^{s=\infty}\left\{\frac{\sin(n-s)\alpha}{n-s} + \frac{\sin(n+s+1)\alpha}{n+s+1}\right\}\left(\frac{a}{r}\right)^{s+1} P_s.$$

If we differentiate with respect to α, it is not difficult to show that the second series,

$$= a\int_0^\alpha \cos\frac{2n+1}{2}\alpha\left\{\cos\frac{\alpha}{2}\left[(p+q)^2-4c^2\right]^{\frac{1}{2}} + \sin\frac{\alpha}{2}\left[4c^2-(p-q)^2\right]^{\frac{1}{2}}\right\}\frac{d\alpha}{pq}$$

$$= a\int_0^\alpha \cos\frac{(2n+1)\alpha}{2}\, U\, d\alpha \text{ say,}$$

where p and q have the same meaning as in Art. 6, and c is the radius of the rim of the bowl. Hence

$$V = \frac{ma}{\pi}\int_0^\alpha U\,\Sigma_1^\infty \frac{1}{n}\left(\frac{f}{a}\right)^n \cos\frac{2n+1}{2}\alpha\, d\alpha$$

$$= \frac{ma}{\pi}\int_0^\alpha U\left\{\cos\frac{\alpha}{2}\log\frac{a}{a^2-2af\cos\alpha+f^2} - 2\sin\frac{\alpha}{2}\tan^{-1}\frac{f\sin\alpha}{a-f\cos\alpha}\right\}d\alpha$$

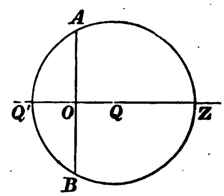
$$\text{.................(33).}$$

14. When the bowl degenerates either into a disc or screen, V will be the potential of the disc or screen when under the influence of an infinite straight line positively electrified with line density m, and extending from the image of the source to infinity, and

$$\Omega = -\frac{dV}{dz}.$$

We shall verify this result in the case of a disc.

Let Q be the source, Q' its image; then the potential (Art. 5) due to a charge m at Q', at all points within the space bounded by sphere AZB and the disc AB, is



$$-P = \frac{m}{\pi}\left\{\frac{1}{Q'P}+\frac{1}{QP}\sin^{-1}\frac{AB\,.\,QP}{QA\,(AP+BP)} - \frac{1}{Q'P}\sin^{-1}\frac{AB\,.\,Q'P}{QA\,(AP+BP)}\right\} \ldots (34).$$

Let (z, ρ) be cylindrical coordinates of P, $OQ = f$, and let

$$F(z, f) = \frac{1}{(f^2+\rho^2)^{\frac{1}{2}}}\sin^{-1}\frac{AB\,(f^2+\rho^2)^{\frac{1}{2}}}{(f^2+c^2)^{\frac{1}{2}}\,(AP+BP)}.$$

Then, if we treat $\sqrt{(f^2+c^2)}$ or QA as constant, whilst differentiating with respect to f,

$$-P = \frac{m}{Q'P} - \frac{2m}{\pi}\left(z\frac{dF}{df}+\frac{z^3}{6}\frac{d^3F}{df^3}\right).$$

Therefore, at the disc,

$$-\frac{d^2P}{dz^2} = m\left(\frac{d}{dz}\right)^2_{z=0}\frac{1}{Q'P} - \frac{4m}{\pi}\left(\frac{d^2F}{df\,dz}\right)_{z=0}$$

$$= -m\frac{d}{df}\frac{f}{(f^2+\rho^2)^{\frac{3}{2}}},$$

since F contains even powers of z only. Hence, if

$$V = \int_f^\infty P\,df.$$

Then, at the disc, $-\dfrac{d^2V}{dz^2} = \dfrac{d\Omega}{dz} = \dfrac{mf}{(f^2+\rho^2)} = -\dfrac{d\Phi}{dz}$,

therefore $$\phi = -\int^\infty \frac{dP}{dz}df + \Phi \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (35).$$

15. *To find the velocity potential due to the motion of the bowl in an infinite liquid.*

1st. Consider the case of motion parallel to the axis.

If the liquid were flowing from right to left past the bowl, the velocity at infinity being equal to C, then

$$\Phi = -Cz,$$

$$\phi = \Omega_z - Cz,$$

and
$$\frac{d\Omega_z}{dr} = C\cos\theta$$
at the surface.

Hence, if the bowl be moving parallel to its axis with velocity C,

$$\phi = \Omega_z.$$

Now, by (23), $\quad V = -\tfrac{1}{2}Ca^2\cos\theta$

at the surface. V is therefore the potential of the induced charge, when the bowl is placed in a uniform field of force parallel to its axis, and its value is given by (15); whence

$$\phi \text{ or } \Omega_z = -\frac{1}{a}\frac{d(Vr)}{dr}.$$

2ndly. Let the bowl be moving perpendicular to its axis with velocity A, and let the plane from which the *angle* ϕ is measured contain the direction of motion; then

$$\frac{d\phi}{dr} = A\cos\phi\sin\theta,$$

therefore
$$V = -\tfrac{1}{2}Aa^2\cos\phi\sin\theta$$

at the surface. V is therefore the potential of the induced charge, when the bowl is placed in a uniform field of force perpendicular to a plane containing its axis, and its value is given by (20); whence ϕ can be found.

3rdly. Let the bowl be rotating about an axis.

It is clear that, if the bowl were rotating about an axis through the centre, the bowl would simply cut its way through the liquid without producing any motion. Now, a rotation about any other axis is equivalent to a rotation about a parallel axis through the centre, together with a velocity of translation perpendicular to the plane containing the centre of the bowl, and the original axis of rotation; hence the

motion of the liquid due to the rotation of the bowl is equivalent to that due to a properly chosen motion of translation.

The foregoing results could only be approximately realised in practice, owing to the fact that, so far as our knowledge extends, there is no such thing as a perfect fluid in nature. It will be noticed that the velocity of the liquid given by the preceding formulæ is infinite at the rim of the bowl, so that the liquid would be torn asunder, and the conditions of continuity violated. The result would be, in the case of an ordinary liquid, that vortex rings would be produced, which would probably be circular when the motion is symmetrical about the axis of the bowl, and this would materially alter the character of the motion.

Note on the Porism of the Inscribed and Circumscribing Polygon.

By L. J. Rogers, Balliol College, Oxford.

[*Read June* 11*th*, 1885.]

Let S and S' be two conics, and let ξ_1, ξ_2, ξ_3 be the roots of the equation obtained by equating the discriminant of $\xi S + S'$ to zero.

It is then possible to express concisely in terms of these roots the condition that a polygon of n sides should be found which may circumscribe S' and be inscribed in S.

The conditions are as follows :—

First, let the roots ξ_1, ξ_2, ξ_3 be all positive and in descending order of magnitude.

Then the condition for a polygon of n sides is that

$$\xi_3 = \xi_1 \operatorname{cn}^2 \frac{2K}{n}, \quad \text{mod.} \sqrt{\frac{\xi_2 - \xi_3}{\xi_1 - \xi_3} \cdot \frac{\xi_1}{\xi_2}}.$$

Secondly, let one of the roots be negative, viz. ξ_3, and let $\xi_1 < \xi_2$.

Then the condition is, if n be even, that

$$\xi_2 k^2 \operatorname{cn}^2 \frac{2K}{n} = \xi_1 \operatorname{dn}^2 \frac{2K}{n}, \quad \text{mod.} \sqrt{\frac{\xi_2 - \xi_3}{\xi_1 - \xi_3} \cdot \frac{\xi_1}{\xi_2}};$$

but, if n be odd, the condition fails.

These two cases will include every possible case in which the roots of the cubic are all real. The case in which the roots are imaginary will be considered later.

1. Let all the roots be positive.

The discriminating cubic may be evidently written in the form

$$\Delta\,(\xi-\xi_1)\left(\xi-\xi_1\,\frac{\mathrm{cn}^2 v}{\mathrm{dn}^2 v}\right)(\xi-\xi_1\,\mathrm{cn}^2 v),$$

according to the above-mentioned conditions.

Now, Prof. Cayley has proved (see Salmon's *Conic Sections*, § 376, Note) that the required conditions depend upon the vanishing of certain determinants, which are clearly seen to vanish also when all the roots of the discriminating cubic are multiplied by the same quantity.

Hence, if all the roots are of the same sign, we get the same conditions as if the cubic were

$$(\xi-1)\left(\xi-\frac{\mathrm{cn}^2 v}{\mathrm{dn}^2 v}\right)(\xi-\mathrm{cn}^2 v).$$

Now, this cubic corresponds to the case of two confocals

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

and

$$\frac{x^2\,\mathrm{cn}^2 v}{a^2\,\mathrm{dn}^2 v}+\frac{y^2}{b^2}\,\mathrm{cn}^2 v=1,$$

for which Mr. Griffiths has proved (*Proc. Lond. Math. Soc.*, Vol. XIV., p. 47) the condition for a polygon of n sides is that

$$v=\frac{2K}{n},\quad \mathrm{mod.}\ \sqrt{\frac{a^2-b^2}{a^2}}.$$

Hence we have

$$\xi_3=\xi_1\,\mathrm{cn}^2\,\frac{2K}{n},\qquad \mathrm{mod.}\ \sqrt{\frac{\xi_2-\xi_3}{\xi_1-\xi_3}\cdot\frac{\xi_1}{\xi_2}},$$

the modulus being obtained by eliminating v.

Since $\xi_2<\xi_1$, the modulus is always <1.

2. Secondly, let the roots be not all the same sign.

Then, if $\xi_2>\xi_1$, and ξ_3 be negative, the discriminating cubic can be

written

$$\Delta\,(\xi-\xi_1)\left(\xi-\xi_1\frac{\mathrm{dn}^2 v}{k^2\,\mathrm{cn}^2 v}\right)\left(\xi+\xi_1\frac{\mathrm{dn}^2 v}{k^2\,\mathrm{sn}^2 v}\right),$$

which is derived from the other by writing $v+iK'$ for v.

Hence, if the roots be not all the same sign, we get the same conditions as if the cubic were

$$(\xi-1)\left(\xi-\frac{\mathrm{dn}^2 v}{k^2\,\mathrm{cn}^2 v}\right)\left(\xi+\frac{\mathrm{dn}^2 v}{k^2\,\mathrm{sn}^2 v}\right).$$

This cubic corresponds to the case of the confocal ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

and hyperbola
$$\frac{x^2\,\mathrm{dn}^2 v}{a^2k^2\,\mathrm{cn}^2 v}-\frac{y^2\,\mathrm{dn}^2 v}{b^2k^2\,\mathrm{sn}^2 v}=1.$$

Since the latter equation may be written

$$\frac{x^2}{a^2}\,\frac{\mathrm{cn}^2(v+iK')}{\mathrm{dn}^2(v+iK')}+\frac{y^2}{b^2}\,\mathrm{cn}^2(v+iK')=1,$$

we have for the required condition, following Mr. Griffiths' proof, that

$$v=\frac{2K}{n}-iK'+\frac{2miK'}{n}.$$

Now v cannot be real unless n be even, and in this case

$$v=\frac{2K}{n}.$$

Hence, if ξ_1, ξ_2 be the roots of like signs, and ξ_2 be numerically greater, the condition is that

$$\xi_2 k^2\,\mathrm{cn}^2\frac{2K}{n}=\xi_1\,\mathrm{dn}^2\frac{2K}{n},\quad \text{mod.}\ \sqrt{\frac{\xi_2-\xi_3}{\xi_1-\xi_3}\cdot\frac{\xi_1}{\xi_2}}.$$

In this case also the modulus is always <1.

The condition for a polygon of n sides has therefore been completely determined in the case of imaginary roots, except in the second case when n is odd. It appears, however, that in such a case no such polygon can be found.

3. Thirdly, let two of the roots be imaginary.

In this case we must consider the two conics whose discriminating

cubic is $(\xi-1)(\xi-r\cos\theta-ri\sin\theta)(\xi-r\cos\theta+ri\sin\theta)$.

This cubic corresponds to the parabola (inscribed conic)

$$y^2 = 4x,$$

and the circle (circumscribing conic)

$$\left(x+\frac{2}{r}\cos\theta\right)^2+y^2=\frac{4}{r^2}.$$

These conics intersect in two, and only two, real points.

Let (x, y) be any point P on the circle, and tangents PT, PT' be drawn to the parabola, touching it in T and T'.

Then, if m_1, m_2 be the cotangents of their inclinations to the axis of x; m_1, m_2 are the roots of

$$y=\frac{x}{m}+m, \quad \text{or} \quad m^2-my+x=0,$$

therefore $$m_1+m_2=y, \quad m_1m_2=x.$$

Assume that $$m_1 = e\,\mathrm{cn}\,(u+v),$$

$$m_2 = e\,\mathrm{cn}\,(u-v),$$

where e, v, k have to be determined in terms of r and θ, and u depends upon the position of P only.

For brevity, let us put

$$\mathrm{sn}\,u=a, \quad \mathrm{sn}\,v=b, \quad \cos\theta=rl.$$

Then $$x=e^2\,\mathrm{cn}\,(u+v)\,\mathrm{cn}\,(u-v)=e^2\,\frac{1-a^2-b^2+k^2a^2b^2}{1-k^2a^2b^2},$$

$$y=e\left\{\mathrm{cn}\,(u+v)+\mathrm{cn}\,(u-v)\right\}=2e\,\frac{\sqrt{1-a^2}\,.\,\sqrt{1-b^2}}{1-k^2a^2b^2},$$

and the equation to the circle is

$$\left(e^2\,\frac{1-a^2-b^2+k^2a^2b^2}{1-k^2a^2b^2}+2l\right)^2+4e^2\,\frac{(1-a^2)(1-b^2)}{(1-k^2a^2b^2)^2}=\frac{4}{r^2}.$$

In order that this relation may hold for all positions of P, it is necessary that this last equation should hold identically for all values of a.

Equating coefficients of a^0, a^2 and a^4 to zero, we get three equations which give us e, k, b in terms of r and θ,

These three equations are

$$\{e^2(1-b^2)+2l\}^2+4e^2(1-b^2) = \frac{4}{r^2},$$

$$(k^2b^2e^2-e^2-2lk^2b^2)^2 = \frac{4k^4b^4}{r^2},$$

$$2e^2(1-b^2)-\{e^2(1-b^2)+2l\}(k^2b^2e^2-e^2-2lk^2b^2) = \frac{4k^2b^2}{r^2},$$

which, after some reductions, give

$$e = k\frac{\operatorname{sn} 2v}{\operatorname{dn} 2v},$$

$$r = \frac{2\operatorname{dn} 2v}{1+\operatorname{cn} 2v},$$

$$r\cos\theta = \frac{k^2k'^2\operatorname{sn}^4 v-\operatorname{dn}^4 v}{\operatorname{cn}^2 v}.$$

Eliminating v from the last two, we get

$$k = \cos\frac{1}{2}\tan^{-1}\frac{r\sin\theta}{1+r\cos\theta},$$

which is always real. We also get

$$r\cos^2\frac{\theta}{2} = \frac{k^2k'^2\operatorname{sn}^4 v}{\operatorname{cn}^2 v},$$

$$r\sin^2\frac{\theta}{2} = \frac{\operatorname{dn}^4 v}{\operatorname{cn}^2 v},$$

which give k more readily.

Hence, we see that in passing from T to T' the parameter of T is increased by $2v$, where v only depends upon the circle, and not upon the position of the point P.

Hence, as before, the required condition for a polygon of n sides is

that $$2vn = 4K,$$

or $$v = \frac{2K}{n}.$$

Hence, for any two conics, the roots of whose discriminating cubic

are proportional to 1, $re^{\theta i}$, $re^{-\theta i}$, the required condition is that

$$r = \frac{2\,\mathrm{dn}\,\frac{4K}{n}}{1+\mathrm{cn}\,\frac{4K}{n}}, \quad \text{mod. } \cos\frac{1}{2}\tan^{-1}\frac{r\sin\theta}{1+r\cos\theta}.$$

It can be easily verified that for a triangle

$$r = 4\sin^2\frac{\theta}{2},$$

and for a quadrilateral $\qquad r+2\cos\theta = 0.$

On the Ideal Geometrical Form of Natural Cell-Structure.

By Mrs. Bryant, D.Sc.

[*Read March* 12*th*, 1885.]

Ideal natural cell-structure is not necessarily regular, in the strict geometrical sense of the word, as, by convention, it is used to denote solids with identically equal faces and solid angles. A cell-structure regular, in this conventional sense, would clearly consist of cells cubical in form; and such structure, as we shall see, is not natural.

The form of a natural structure is a logical result of its mode of genesis, and that form is ideal of which the mode of genesis is perfectly regular. Moreover, the original cell is spherical in form. Hence the solution of our problem turns upon the double question :—

1. If space is filled with equal spheres, and this space-ful of spheres is then crushed together symmetrically till the whole becomes a solid mass, what shape does each sphere ultimately assume?

2. If a homogeneous solid has equally efficient centres of excavation or absorption distributed uniformly in it, what is the ultimate form of the cells excavated? it being supposed that, when the excavating or absorbing agents cease their work, the walls of the cells are uniform in thickness, *i.e.*, the excavation is complete.

The second question is manifestly the counterpart of the first, and is answered in the answer to the first.

Our first step must be to determine the mode of arrangement of the

spheres which, by the terms of the question, fill space in a natural, as well as a regular, manner. Now, there are three conceivable ways in which the spheres can be arranged regularly. Only one of these is, however, a natural, because a *mechanically stable*, arrangement, *i.e.*, the one in which the conditions of *maximum density* and *maximum stability* are fulfilled,—in which, therefore, the centres of all the spheres that touch any given sphere are at the minimum mutual distance of the common diameter.

The three geometrically possible arrangements may be conceived as follows, for convenience in deducing the corresponding forms of the crushed spheres:—

(1) The central sphere touches the *six* faces of a cube at their mid-points, and six surrounding spheres at the same points. No two of these six touch each other; the centres of any adjacent two are at a distance equal to $\sqrt{2}$ of the diameter, and they are not, therefore, situated similarly with respect to the central sphere and to one another. This may be called a cubical arrangement, and, if crushed, will yield cubes filling space without interstices.

(2) The central sphere passes through, and touches *eight* surrounding spheres at, the eight vertices of a cube. No two of these eight surrounding spheres touch each other, and the centres of any adjacent two are at a distance equal to $\frac{2}{\sqrt{3}}$ of the diameter. This arrangement is, therefore, denser than the first; but, like it, is deficient in the mutual support of its parts, and in the more perfect symmetry which belongs, as we shall see, to the third. If crushed, it will yield octohedra with tetrahedral interstices, these together filling space, as is well known; but, since there are interstices, the crushing cannot be complete. The proof of this need not detain us here.

(3) The central sphere touches the *twelve* edges of a cube, and twelve surrounding spheres, at the mid-points of the edges. The radius of the spheres is in this case equal to the semi-diagonal of the cube's face, and this is clearly equal to the distance between the points of contact of two adjacent spheres with the central sphere. The distance between their centres is, therefore, by similar triangles twice, the radius. Hence they touch; and thus the twelve surrounding spheres are in contact with each other three-and-three about the eight corners of the cube, while they are in contact four-and-four about the six faces of the cube.

This arrangement is of *maximum density*, since the surrounding spheres have their centres at the *minimum distance* of the common diameter. It is also the arrangement of *maximum stability*, because

the *mutual support* of its parts is the greatest possible, as each of the surfaces bounding intervening spaces touches all the others. This is, therefore, the natural arrangement; and, as a matter of fact, if a quantity of shot be thrown into a box and shaken about freely, it will arrange itself in this way. Hence, too, the natural form of a pile of balls is a pyramid on a regular hexagonal, or, which is the same thing, an equilateral triangular, base. Piles on square bases are also common; but, as can easily be seen, the elementary arrangement is exactly the same; in the triangular pile, a face of the elementary tetrahedron of adjacent centres is horizontal, and in the square pyramidal pile, an edge.

While the mechanical instability of any but this dodecahedral arrangement of spheres determines it as the natural arrangement in a space-ful of spheres, its property of maximum density is a reason for considering it of fundamental importance in considering the natural mode of distribution of excavators or absorbents in a solid, since by it the maximum of excavation or absorption in a given space can be secured. Moreover, it is, as we have seen, more perfectly symmetrical than any other arrangement.

I. Considering, first, the case of the spheres to be symmetrically crushed together, let us limit our attention to the central sphere, which touches the twelve adjacent spheres at the mid-points of the twelve edges of a cube, its intermediate portions bulging out through the six faces of the cube. When the spheres are crushed together, these twelve points of contact move inwards along the radii, and the six intermediate portions are squeezed out into the over-arching spaces which lie between the points of contact of the surrounding spheres. Since there are four spheres round every face, these portions will be squeezed into four-sided pyramids, the faces of each being evidently conterminous with those of the adjacent pyramid, both being the ultimate position of the original plane of contact. Each, therefore, makes half a right angle with the face of the cube, the sum of the two being supplementary to the angle between the cube's faces. Hence, in the final position, we have the twelve points of contact represented by the mid-points of the edges of a smaller cube, and the intermediate portions heaped up into six square pyramids on the faces of the cube, whose faces make half a right angle with those of the cube. The form thus generated is a solid with twelve rhombic faces, the well-known rhombic dodecahedron.

There will be no intervening spaces in the mass of solids when the spheres are completely crushed together; because dodecahedra of this kind can be packed so as to fill space without interstices. To prove

this, it is convenient to consider the dodecahedron as built up by dividing a cube into the six equal pyramids which have their vertices at the intersection of its diagonals, and placing these on the faces of an equal cube. The twenty-four faces of the figure reduce, as in the above, to twelve; since the diagonal plane of a cube makes half a right-angle with the cube's face, and hence two adjacent faces of any two pyramids lie in one plane and form a rhombus.

Six of the solid angles are enclosed by four planes, perpendicular to each other, two and two, since they were originally the diagonal planes of a cube. Therefore, four of the solids superposed on these faces at any vertex will just leave room for the vertex of another solid in the remaining space.

The other eight solid angles are situated at the vertices of the original cube, being enclosed by three plane angles, which are the obtuse angles of the rhombi, and are easily seen to be equal to those between the opposite diagonals of the cube. Now, by parallels, the diagonal of the dodecahedron through one of these vertices makes, with an edge of the dodecahedron, an angle equal to that which it makes with the opposite diagonal. Hence, the diagonal makes an exterior angle with each of the edges equal to the obtuse angle of the faces. When, therefore, three solids are superposed on the faces at such a vertex, their edges, meeting in that vertex, coincide along the diagonal of the central solid.

Space can, therefore, be filled with rhombic dodecahedra; and the crushed spheres, consequently, form a mass without intervening spaces.

We should expect to find this dodecahedral form in nature wherever originally spherical cells, packed together in the most natural or in the closest manner, have been subjected to uniform and complete pressure. The two conditions, (1) of initial symmetrical arrangement, and (2) of complete symmetrical pressure, are probably seldom fulfilled simultaneously, as a matter of fact; and so, nature transgresses her own ideal of naturalness in this as in other respects. In the centre of a mass of soap-bubbles their chance of fulfilment is perhaps at its best: but, in the fact that the bubbles tend to stick to one another, there is a disturbing element, even in the centre of the mass; and the difficulty of seeing the form within the mass is great.

2. Reverting now to our second question, it is, as before remarked, evident that the structure produced by complete and symmetrical pressure of spherical cells, symmetrically distributed in a space, is of the same form as that produced by the complete and symmetrical activity of equally efficient centres of excavation or absorption,

symmetrically distributed in a solid. The dodecahedral form is, therefore, the ideal natural form of cell-structure which is caused by absorption, as organic structures often are, or by excavation as in the case of the honey-comb cell.

In organic structure it is not likely that the ideal conditions are ever completely fulfilled, and frequently the actual conditions are quite different. But, as regards the honey-comb, we might reasonably expect that the bees, who are the cell-excavators, should by natural instinct distribute themselves *as densely as possible*, and with a considerable degree of regularity, and that their activities should be equal and symmetrical about the working parts of their bodies. The facts confirm this reasonable expectation. The bees distribute themselves, with apparent uniformity, at the two sides of a homogeneous cake of wax which has been previously deposited. In it they excavate cells, at doubtless uniform rates of work, and continue excavating till their work is as complete as possible, and the walls of the cells therefore of uniform thickness. Meanwhile, the excavated wax is used to build up higher the open cell walls. Hence, the cells ought to be elongated rhombic semi-dodecahedra; and this is just what they are, the axis of the cell corresponding to a diagonal of the primary cube, and the apex being one of the trihedral vertices of the dodecahedron. Each face at the apex fits exactly against one face of a cell in the opposite system. Each cell, therefore, is in contact with three cells of the opposite system.

It follows, from this last mentioned fact, that the bees must distribute themselves with maximum density, not only on each side separately, but on the two considered jointly. This, as a case of instinct, is certainly remarkable, but the possibilities of trial and error are sufficient to account for it. It is not unreasonable to expect that the bees should learn how to employ the largest possible number of themselves on a piece of wax to be excavated, this being a thing which they would *naturally try to do;* though it would be strange, in comparison, if they tried to effect those other two ends, of maximum economy in wax, and maximum strength of structure, which, as a matter of fact, they do effect. Whatever it is natural that they should try to do, it is natural that they should succeed in doing. And so, it is no less intelligible than remarkable, that our one clear example of nature fulfilling her own ideal of a natural cell-structure, should be the work of simple animal instinct in the construction of the honey-comb.

The following presents were received in the Recess:—

Carte-de-visite likeness of the Rev. J. J. Milne, M.A.

"Educational Times," for June.

"Physical Society—Proceedings," Vol. vi., Part 4, Jan., Feb., 1885.

"Royal Dublin Society—Scientific Proceedings," Vol. iv. (N.S.) Parts 5 and 6; "Scientific Transactions," Vol. iii. (Ser. ii.), Parts 4, 5 and 6.

"A Memoir on Biquaternions," by Arthur Buchheim (from the "American Journal of Mathematics,") Vol. vii., No. 4, Quarto.

"Johns Hopkins University Circulars," Vol. iv., No. 39.

"Sur les Isométriques d'une Droite par rapport à certains systèmes de courbes planes" ("Bulletin de la Société Mathématique de France," T. xiii., 1885).

"Sur une Transformation polaire des Courbes planes" ("Jornal de Sciencias Mathematicas e Astronomicas").

"Note sur les Raccordements paraboliques," ("Mathesis," T. v., 1885), par M. M. d'Ocagne: from the Author.

"Annali di Matematica," Serie ii^a, Tomo. xiii., Fasc. 2° (Giugno, 1885); Milano.

"American Journal of Mathematics," Vol. vii., No. 4; Baltimore, 1885.

"The Mathematical Theory of Electricity and Magnetism," by H. W. Watson, D.Sc., F.R.S., and S. H. Burbury, M.A.; Vol. i., Electrostatics (Clarendon Press Series, 1885).

"Elements of Projective Geometry," by Luigi Cremona, translated by C. Leudesdorf, M.A. (Clarendon Press Series, 1885): presented by the Clarendon Press authorities.

"Jahrbuch über die Fortschritte der Mathematik, &c.," B. xiv., H. 3; Berlin, 1885.

"Beiblätter zu den Annalen der Physik und Chemie," B. ix., St. 5.

"Bulletin de la Société Mathématique de France," T. xiii., No. 3.

"Bulletin des Sciences Mathématiques," T. ix., June, 1885.

"Proceedings of the Royal Society," Vol. xxxviii., Nos. 237, 238.

"Educational Times," for July to October.

"Problems on the Motion of Atoms," by J. K. Smythies, 4to; London, 1885.

"Greek Geometry, from Thales to Euclid," Part vi., by Prof. Allman, 8vo; Dublin, 1885.

"Instantaneous Arithmetic: a New System of Mental Calculation," by S. Hunter, 8vo; London, 1885.

"Johns Hopkins University Circulars," Vol. iv., Nos. 40, 41.

"Bulletin de la Société Mathématique de France," T. xiii., Nos. 4, 5, 6.

"Bulletin des Sciences Mathématiques," Sér. iii., T. ix.; July to October, 1885.

"Crelle," Bd. xcviii., Heft 4; Bd. xcix., H. 1.

"Acta Mathematica," vi., 1, 2, 3, 4, and vii., 1.

"Beiblätter zu den Annalen der Physik und Chemie," B ix., St. 6, 7, 8; 1885.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. vi., No. 2; 1885.

"Die n-dimensionalen Verallgemeinerungen der fundamentalen Anzahlen unseres Raums," von H. Schubert; 8vo pamphlet.

"Die n-dimensionale Verallgemeinerung der Anzahlen für die vielpunktig berührenden Tangenten einer punktallgemeinen Fläche m-ten Grades," von H. Schubert; 8vo pamphlet. (These two from "Math. Annalen," Bd. xxvi.)

"Atti della R. Accademia dei Lincei—Rendiconti," Vol. i., F. 12—20.

"Osservazioni Meteorologiche fatte al R. Osservatorio del Campidoglio dal Luglio al Dicembre 1884," 4to; Rome, 1885.

"Uber die elliptischen Normalcurven der N-ten Ordnung und zugehörige Modulfunctionen der N-ten Stufe," von F. Klein (two copies, 4to and large 8vo); Leipzig, 1885.

"Rendiconti del Circolo Matematico di Palermo," Marzo, 1884, to Marzo, 1885, 4to.

"Vierteljahrschrift der Naturforschenden Gesellschaft in Zürich," Vols. XXVI. to XXIX., 1881—1884.

"Bulletins de l'Académie Royale des Sciences, des Lettres, et des Beaux Arts de Belgique," Ser. III., T. VI., VII., VIII.; 1883, 1884.

"Annuaires de l'Académie Royale des Sciences, des Lettres, et des Beaux Arts de Belgique," 1884 and 1885.

Three Russian Books.

"The Mathematical Visitor," edited and published by Artemas Martin, M.A., Vol. II., No. 2 (January, 1883, but contains matter up to date July 16, 1885, the publication having been delayed in consequence of ill-health of Editor).

"Annali di Matematica pura ed Applicata," Serie II., Tomo XIII., Fasc. 3; Settembre, 1885.

"American Journal of Mathematics," Vol. VIII., No. 1; September, 1885.

"Proceedings of the Physical Society of London," Vol. VII., Part I.; July, 1885.

"Proceedings of the Canadian Institute," 3rd Series, Vol. III., Fasc. No. 2; Toronto, July, 1885.

"Transactions of the Connecticut Academy of Arts and Sciences," Vol. VI., Part II.; Newhaven.

Papers by W. Woolsey Johnson, reprinted from the "American Journal of Mathematics," 4to, Vol. VII.

"Telegraphic Determination of Longitudes in Mexico and Central America, and on the West Coast of South America," 4to; Washington, 1885.

"Archives Néerlandaises des Sciences Exactes et Naturelles," T. XX., Liv. 1 and 2; Harlem, 1885.

"Berichte über die Verhandlungen der K. Sächischen Gesellschaft der Wissenschaften zu Leipzig," 1884, I., II.; 1885, I., II.

"Die bei der Untersuchung von Gelenkbewegungen Anzuwendende Methode erläutert am Gelenkmechanismus des Vorderarms beim Menschen," von Wilhelm Braune in verbindung mit Otto Fischer, large 8vo; Leipzig, 1885.

"Ueber die Methode der Richtigen und Falschen Fälle in anwendung auf die Massbestimmungen der Feinheit oder extensiven Empfindlichkeit des Raumsinnes," von G. Th. Fechner, large 8vo; Leipzig, 1884.

"Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin," I. to XXXIX.; Berlin, 1885.

APPENDIX.

Mr. Mukhopâdhyây's "Theorem in Plane Conics" (p. 7) forms Question 8337, in the "Educational Times" for November, 1885.

The "Group of Circles" considered by Mr. Tucker (p. 44) was the following:—It is well-known that, if POP' is a diameter of the circumcircle of a triangle ABC, the two Simson-lines corresponding to P and P' intersect at right angles on the Nine-point Circle, and the envelope of the Simson-lines is a Tricusp.* The circles, considered in the communication, were obtained by taking points l, m, n, &c., on the perpendiculars PL, PM, PN, &c., such that $lL=K.PL$, $mM=K.PM$, $nN=K.PN$, &c.; then lmn is parallel to the Simson-line LMN, and the corresponding line $l'm'n'$ is parallel to the Simson-line $L'M'N'$: lmn, $l'm'n'$ intersect at right angles on a circle. The properties of this system of circles were worked out on the lines of the well-known results and shown to be quite analogous.

Mr. Heppel's communication (p. 44), was on the "Reduction of the General Equation of the Second Degree." The result arrived at was: Suppose the ellipse $ax^2+2hxy+by^2+c'=0$ transforms to $\dfrac{x^2}{p^2}+\dfrac{y^2}{q^2}=1$. Then, turning back again through an angle $-\theta$, the latter equation becomes

$$(q^2\cos^2\theta+p^2\sin^2\theta)\,x^2+(q^2-p^2)\sin 2\theta\, xy+(q^2\sin^2\theta+p^2\cos^2\theta)\,y^2-p^2q^2=0,$$

where q^2-p^2 is essentially negative.

Therefore, comparing with $ax^2+2hxy+by^2+c'=0$, θ is $>$ or $<\dfrac{\pi}{2}$, as $\dfrac{h}{c'}$ is negative or positive.

Next, let the hyperbola $ax^2+2hxy+by^2+c'=0$ transform to

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* See *Educational Times* (Quest. 1649), Feb., 1865; *Reprint*, Vol. III., pp. 58, 97; Vol. IV., pp. 13, 27, 74, 81, 94; *Lady's and Gentleman's Diary*, 1861—1863. The property was originally published by Steiner, without demonstration, in *Crelle*, Vol. LIII. Dr. Casey, in his "Sequel to Euclid" (3rd ed., 1884, p. 174, Ex. 31), states that the property was communicated to him by a friend, and M. N. Goffart also seems to have thought the property was a new one (see *Nouvelles Annales de Math.*, Quest. 1473, the solution being given on p. 397 of the No. for August, 1884).

$\dfrac{x^2}{p^2} - \dfrac{y^2}{q^2} = 1$, then, transforming back again, we get similarly $\theta >$ or $< \dfrac{\pi}{2}$, as $\dfrac{h}{c'}$ is positive or negative. Hence, in the ellipse, major axis is in first quadrant if h/c' is positive; in the hyperbola, the real axis is in the first quadrant if h/c' is negative.

The Rev. R. Harley's remarks on "Criticoids" (p. 61) will be found in a paper contributed by him to the Royal Society (*Proc. of Royal Society*, Vol. XXXVIII., No. 235, pp. 45—57), entitled, "Professor Malet's Classes of Invariants identified with Sir James Cockle's Criticoids."

Captain Macmahon's communication (p. 61) will be found in the *Messenger of Mathematics*, Vol. XIV., No. 11 (March, 1885, p. 164); in which volume also is printed Mr. Buchheim's Note (p. 61), No. 9 (January, 1885, p. 143).

Mr. Griffiths sends the following "Note on Partial Multiplication for Modulus $1/\sqrt{2}$":—

The following is an example of the partial multiplication noticed on p. 106, viz., $n = 5$, and

$$y = \frac{x\left\{1+2i-(1+2i)\,x^2+\dfrac{i}{2}\,x^4\right\}}{1+(i-2)\,x^2+\left(1-\dfrac{i}{2}\right)x^4}. \qquad (i=\sqrt{-1}).$$

This gives $$\frac{dy}{\sqrt{1-y^2\,.\,1-\frac{1}{2}y^2}} = (1+2i)\,\frac{dx}{\sqrt{1-x^2\,.\,1-\frac{1}{2}x^2}},$$

i.e., $$v = (1+2i)\,u\,; \qquad \left(\text{mod. } \frac{1}{\sqrt{2}}\right).$$

Here $y = 1$ when $x = 1$, and $y = -\sqrt{2}$, when $x = \sqrt{2}$;

i.e., $\quad \text{sn}\,(K+2iK') = 1$, and $\text{sn}\,(1+2i)(K+iK') = -\sqrt{2}$,

or, since $K' = K$, $\quad \text{sn}\,(-K+3iK') = -\sqrt{2}$.

Referring to p. 108, it may be remarked that, if we put $\Lambda_2 = \Lambda_2' = K = K'$, we have $M_2 = -(1+2i)$.

This is the case of $-y =$ same function of x as above.

As regards the general theory of composition, I would observe, that (1) all the transformation equations of an even order of the form $y =$ rational function of x, can be derived from the formula

$$y = \sin\,(L+A+B+\ldots),$$

as I hope to show in a future note; (2) Landen's equation can be introduced into Jacobi's transformations of an odd order. In fact, the first transformation is

$$y = \sin\left\{-\theta+2L+A+B+\dots\right\},$$

if we write

$$\sin\theta = x,\quad \sin L = \frac{(1+a')\,x\sqrt{1-x^2}}{\sqrt{1-a^2x^2}},\quad \cos L = \frac{1-(1+a')\,x^2}{\sqrt{1-a^2x^2}};$$

$$\cos A = \frac{1-(1+a'^2)\,x^2}{1-a^2x^2},\quad \sin A = \frac{2a'x\sqrt{1-x^2}}{1-a^2x^2},\ \&c.,$$

$$a^2+a'^2 = 1 = a^2+a'^2 = \dots,$$

$$a = k\,\mathrm{sn}\,\frac{2K}{n},\quad a' = \mathrm{dn}\,\frac{2K}{n};\qquad a = k\,\mathrm{sn}\,\frac{4K}{n},\quad a' = \mathrm{dn}\,\frac{4K}{n},\ \dots$$

The analogy between elliptic transformations and trigonometrical formulæ is very striking.*

The paper on "Some Properties of the Harmonic Quadrilateral," by Mr. Tucker (p. 184), will, through the courtesy of the Messrs. Hodgson, the proprietors of the *Reprint*, be published for the author, who desired an early appearance for the paper, as a supplement to Vol. XLIV.†

Prof. J. Neuberg has furnished to the author "proofs" of an

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* Mr. Griffiths furnishes the following list of corrections to be made in his paper, pp. 83—108:

p. 84, line 15, in the expressions *for* a and a' *dele* δ.

p. 85, last line, *for* $i\tan(X+X_1+\dots+X_m)$ *read* $\tan(X+X_1+\dots+X_m)$.

p. 86, line 7, *for* $a_1, a_2 \dots a_m$ *read* $b_1, b_2, \dots b_m$.

p. 91, line 10, *read* $2n\Gamma' = NK'$ *for* $2n\Gamma' = NK$.

p. 97, line 16, in the relation between the moduli *read* $\sqrt{k'}$ *for* $\sqrt{k}$ as regards the numerator.

p. 103, line 9 from bottom, in the denominator *read* k^2 *for* k.

p. 104, line 2, *read* k^2 *for* k^4.

p. 104, *for* $\sin X_2$, $\sin X_3$, $\sin X_4$, respectively, the expressions should be

$$\sin X_2 = (1-\sqrt{k})^2\dots\ \text{(rest as printed)},$$

$$\sin X_3 = (1+i\sqrt{k})^2\,\frac{\mathrm{sn}\,u\,(1-k\,\mathrm{sn}^2u)}{1+2i\sqrt{k}\,(1+i\sqrt{k}-k)\,\mathrm{sn}^2u+k^2\,\mathrm{sn}^4u},$$

$$\sin X_4 = (1-i\sqrt{k})^2\,\frac{\mathrm{sn}\,u\,(1-k\,\mathrm{sn}^2u)}{1-2i\sqrt{k}\,(1-i\sqrt{k}-k)\,\mathrm{sn}^2u+k^2\,\mathrm{sn}^4u}.$$

p. 105, line 2 from bottom, *change* M_3 *into* $\frac{n}{M_3}$.

† Advance copies have been forwarded to a large number of the members.

article by him, " Sur le Quadrilatère Harmonique," in recent numbers of *Mathesis* (Oct. and Nov., pp. 202–212, 217–223), in which he works out the results on the lines of his Mémoire " Sur le Centre des Médianes antiparallèles."

Prof. Neuberg has also communicated to the *Reprint* (Vol. XLIII., pp. 81–85) a paper entitled, " Sur les Cercles de Tucker." This is the " group of circles " of *Quarterly Journal of Mathematics* (Vol. XX., No. 77). In this article he also furnishes information upon matters treated of in the Appendix to last Session's volume.

The Rev. T. C. Simmons' communication (p. 262) will appear in the Appendix to Vol. XLIV. of the *Reprint.*

M. Figarié, of Paris, is writing a memoir for the *Journal de Math. Elémentaires*, of M. de Longchamps, on the theorems in Mr. H. M. Taylor's paper (Vol. XV., pp. 122-139), and the circles of the same gentleman, Mr. Tucker, and M. Lemoine.

M. E. Lemoine has communicated a paper, " Sur une Généralisation des Propriétés relatives au Cercle de Brocard et au Point de Lemoine," to the *Nouvelles Annales* (Mai, 1885, pp. 201–223); and M. D'Ocagne publishes, in the August number of the same volume (pp. 360–367), a " Note sur la Symédiane."

We are indebted to Dr. H. Lieber for extracts from Vol. XVI. of his *Zeitschrift f. Math. u. Natur. Unterr.*, bearing upon the geometry of Brocard's circle. The questions will be found on pp. 351–355.

R. T.

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INDEX OF PAPERS, &c.

[*The mark * indicates that the Paper has either been printed in extenso, or an abstract of it given in the Proceedings or in the Appendix.*]

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